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Praca licencjacka

Introduction to elliptic partial differential equations

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Promotor
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Akademia Górniczo–Hutnicza im. Stanisława Staszica w Krakowie
Wydział Matematyki Stosowanej

FORMULARZ OCENY PRACY LICENCJACKIEJ

Imię i nazwisko: Timo Rohner
Nr albumu: 302570
Temat pracy: Introduction to elliptic partial differential equations

dr Witold Majdak

(opiekun)

Kryterium	Waga	Ocena
(M) Merytoryczna ocena pracy	0,7	
(F) Ocena formalnej strony pracy	0,3	
(O) Ocena końcowa (0,7 M+0,3 F)		

Uzasadnienie oceny skrajnej (2,0, 5,0):

Ocena za prezentację pracy licencjackiej	....

Podpis opiekuna:..

dr Adam Wegert

(recenzent)

Kryterium	Waga	Ocena
(M) Merytoryczna ocena pracy	0,7	
(F) Ocena formalnej strony pracy	0,3	
(R) Ocena końcowa (0,7 M+0,3 F)		

Uzasadnienie oceny skrajnej (2,0, 5,0):

Podpis recenzenta:..

Ocena pracy (0,5 O + 0,5 R): ....

Streszczenie

Tematem pracy jest wprowadzenie do podstawowej teorii i metod używanych w eliptycznych równaniach różniczkowych cząstkowych. W pierwszym rozdziale dokonano krótkiego przeglądu równań różniczkowych cząstkowych oraz ich klasyfikacji. Rozdział drugi został poświęcony funkcjom harmonicznym oraz ich własnościom. W trzecim rozdziale podano wyprowadzenia podstawowych rozwiązań dla równań Laplace'a i Poissona. W ostatnim rozdziale zostały zawarte dwa główne twierdzenia dotyczące istnienia rozwiązań eliptycznych równań różniczkowych cząstkowych, mianowicie metoda Perrona oraz istnienie słabych rozwiązań przy użyciu twierdzenia Laxa-Milgrama.

Abstract

This thesis covers the basic theory and methods used in the study of elliptic partial differential equations. The first chapter gives a quick overview of partial differential equations and their classification. The second chapter revolves around harmonic functions and their properties. The third chapter contains a full development of the fundamental solution to the Laplace equation and solution to the Poisson equation. The fourth chapter covers two major existence results for elliptic PDE, namely the Perron method and the existence of weak solutions using the Lax-Milgram theorem.

Słowa klucze

Równania różniczkowe cząstkowe, funkcje harmoniczne, równanie Poissona, funkcja Greena, metoda Perrona, przestrzenie Sobolewa, twierdzenie Laxa-Milgrama, słabe rozwiązania;

Keywords

Partial differential equations, harmonic functions, Poisson equation, Green function, Perron method, Sobolev spaces, Lax-Milgram theorem, weak solutions;

Contents

1 Introduction

This chapter contains a short introduction to partial differential equations, their classification, and introduces the Laplace and the Poisson equation alongside two basic boundary conditions. For a more in-depth overview of partial differential equations, the reader may find the book *Partial Differential Equations* [2] helpful.

1.1 Second order partial differential equations

1.1.1 Definition

Definition 1.1. Let Ω be an open subset of $\mathbb{R}^n$. A second order partial differential equation is of the form

$$F(D^2u, Du, u, x) = 0, \tag{1.1}$$

where $u : \Omega \to \mathbb{R}$ is an unknown function, D^2u the Hessian matrix of u, and Du the gradient of u.

Definition 1.2. A second order PDE is

i) **linear** if equation (1.1) takes the following form:

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x),$$

ii) **semilinear** if the second order terms of equation (1.1) are linear:

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i x_j} + F'(Du, u, x) = f(x),$$

where F' is a first order PDE that is not linear; otherwise the PDE would simply be linear.

iii) **quasilinear** if the coefficients a_{ij} only depend on Du, u and x:

$$\sum_{i,j=1}^{n} a_{ij}(Du, u, x)\frac{\partial^2 u}{\partial x_i x_j} + F'(Du, u, x) = f(x).$$

If all a_{ij} are independent of Du and u then the PDE is not quasilinear. It is either linear or semilinear, depending on the linearity of F'.

iv) **non-linear** if it is none of the above.

Definition 1.3. $u : \Omega \to \mathbb{R}$ is a classical solution to the PDE (1.1) if

i) $u \in C^2(\Omega)$,

ii) equation (1.1) holds for all $x \in \Omega$.

1.1.2 Classification

For a linear or semi-linear PDE we can rewrite equation (1.1) as follows:

$$F(D^2u, Du, u, x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i x_j} + F'(Du, u, x) = 0.$$

We define a matrix $A(x) = \big(a_{ij}(x)\big)_{i,j=1..n}$ and thanks to the C^2 regularity of classical solutions, the matrix D^2u is symmetric. Without loss of generality, we can assume that A is symmetric. In fact, suppose that A is not symmetric and let $\tilde{A} = \frac{1}{2}\big(A + A^T\big)$. By symmetry of D^2u, we deduce that $A \cdot D^2u = \tilde{A} \cdot D^2u$ and thus, we can replace the non-symmetric matrix A with the symmetric matrix $\tilde{A}$. Since A is real symmetric, it is also diagonalizable, allowing for the classification of any semi-linear or linear PDE using only the eigenvalues of the corresponding matrix A.

Definition 1.4. A linear or semi-linear second order PDE is

i) **elliptic** if the eigenvalues of A are all either strictly positive or strictly negative,

ii) **parabolic** if at least one eigenvalue of A is zero,

iii) **hyperbolic** if all eigenvalues of A are non-zero, and exactly $n-1$ are negative, leaving 1 eigenvalue to be positive or conversely $n-1$ positive, leaving 1 eigenvalue to be negative.

1.2 Basic equations

The Laplace equation is one of the most fundamental equations in the study of elliptic PDE.

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0.$$

An extension of the Laplace equation is the Poisson equation,

$$\Delta u = \varphi,$$

where φ is some function.

1.3 Boundary conditions

The study of PDE is not limited to studying problems of varying linearity. A lot of time and energy goes into studying problems with various boundary conditions. One such boundary condition of interest to us, is the Dirichlet boundary condition. We will restrict ourselves to the study of problems with such boundary conditions. Nonetheless, the notions, methods and tools developed in this work are not limited to the study of elliptic PDE with Dirichlet boundary conditions.

$$u(x) = f(x) \text{ on } \partial\Omega, \qquad \text{(Dirichlet boundary condition)}$$
$$\frac{\partial u}{\partial \nu}(x) = f(x) \text{ on } \partial\Omega, \qquad \text{(von Neumann boundary condition)}$$
where $\frac{\partial u}{\partial \nu}(x)$ is the normal derivative of u, f some function, and $\Omega \subset \mathbb{R}^n$.

2 Harmonic functions

The focus of this chapter are harmonic functions and their properties, which will be useful in subsequent chapters, wherein we will solve the Laplace equation, develop the corresponding fundamental solution, develop Green's function to solve the Poisson equation for various boundary conditions and provide a sufficiently general existence result for the Laplace equation. This chapter loosely follows a series of lectures on elliptic equations, held by Professor Vicentiu D. Radulescu in the fall of 2018 at AGH University of Science and Technology [7]. The lectures focused on providing a general overview and most theorems were given without full proofs. Thus, this chapter seeks to complement and extend some of the material covered in the lectures. The author wishes to acknowledge the important contribution of the lectures on the following material.

Remark 2.1. We will make frequent use of the following notation.

i) $B_r(x)$ denotes a ball of radius r, centered at x, i.e.
$$B_r(x) = \{y \in \mathbb{R}^n \mid |x - y| < r\},$$

ii) ω_n denotes the measure of the boundary of a unit ball in the euclidian space $\mathbb{R}^n$, i.e. the measure of $\partial B_1(0)$,

iii) $\int\limits_{\partial\Omega} d\sigma_x$ denotes the surface integral over $\partial\Omega$,

iv) For a set $\Omega \subset \mathbb{R}^n$, we denote the measure of Ω by $|\Omega|$, where unless stated otherwise the measure used is the Lebesgue measure.

v) $\Omega \subset \mathbb{R}^n$ is an open bounded subset of $\mathbb{R}^n$, unless stated otherwise.

Definition 2.2. A function $u \in C^2(\Omega)$ is harmonic if $\Delta u = 0$.

2.1 Mean value properties

Definition 2.3. Given a function $u \in C(\Omega)$, we say that

i) u satisfies the first mean value property in Ω if
$$u(x) = \frac{1}{|\partial B_r(x)|} \int\limits_{\partial B_r(x)} u(y) d\sigma_y \quad \text{for any} \quad \overline{B_r(x)} \subset \Omega,$$

ii) u satisfies the second mean value property in Ω if

$$u(x) = \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} u(y)dy \quad \text{for any} \quad \overline{B_r(x)} \subset \Omega.$$

Remark 2.4. The first and second mean value property are equivalent. This follows from the definition of the mean value properties. We take the first mean value property and rewrite it as follows:

$$u(x)r^{n-1} = \frac{1}{\omega_n} \int\limits_{\partial B_r(x)} u(y)d\sigma_y.$$

After integrating over r, we can simplify by directly integrating over $B_r(x)$ instead of integrating over $\partial B_r(x)$ and subsequently integrating over $[0, r]$. This follows from Fubini's theorem A.5.

$$u(x)\frac{r^n}{n} = \frac{1}{\omega_n} \int\limits_0^r \int\limits_{\partial B_t(x)} u(y)d\sigma_y dt = \frac{1}{\omega_n} \int\limits_{B_r(x)} u(y)dy.$$

The measure of $B_r(x)$ can be written as $|B_r(x)| = \frac{n}{\omega_n r^n}$. Therefore,

$$u(x) = \frac{n}{\omega_n r^n} \int\limits_{B_r(x)} u(y)dy = \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} u(y)dy.$$

Theorem 2.5. *A function $u \in C^2(\Omega)$ satisfies the mean value properties in Ω if and only if u is harmonic in Ω.*

Proof. **Step 1.** Suppose u is harmonic and let

$$\varphi_x(r) : \{r \in \mathbb{R}_{>0} \mid \overline{B_r}(x) \subset \Omega\} \to \mathbb{R},$$

for any $x \in \Omega$ be defined as follows:

$$\varphi_x(r) = \frac{1}{|\partial B_r(x)|} \int\limits_{\partial B_r(x)} u(y)d\sigma_y.$$

We introduce a change of variable $y = x + r\omega$, which yields

$$\varphi_x(r) = \frac{1}{\omega_n} \int\limits_{\partial B_1(0)} u(x + r\omega)d\sigma_\omega.$$

By taking the derivative of φ_x with respect to r, we get

$$\varphi_x'(r) = \frac{1}{\omega_n} \int\limits_{\partial B_1(0)} \nabla u(x + r\omega) \cdot \omega d\sigma_\omega. \tag{2.1}$$

A simple change of variable $\omega = \frac{y-x}{r}$ allows (2.1) to be transformed into a boundary integral. The Gauss divergence theorem A.9 allows for the boundary integral to be transformed into an integral over $B_r(x)$. Namely,

$$\varphi_x'(r) = \frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(x)} \big(\nabla u(y) \cdot \vec{n}\big) d\sigma_y = \frac{1}{\omega_n r^{n-1}} \int\limits_{B_r(x)} \operatorname{div}(\nabla u(y)) dy$$
$$= \frac{1}{\omega_n r^{n-1}} \int\limits_{B_r(x)} \Delta u(y) dy = 0.$$

Therefore, φ_x is constant. By the Lebesgue differentiation theorem A.11, we have

$$\varphi_x(r) = \lim_{t \to 0^+} \varphi_x(t) = \lim_{t \to o} \frac{1}{|\partial B_t(x)|} \int\limits_{\partial B_t(x)} u(y) d\sigma_y = u(x),$$

and thus u satisfies the mean value property.

Step 2. We show that if u satisfies the mean value property, then u is harmonic. We assume by contradiction that u is not harmonic in Ω and that the mean value property holds. If u is not harmonic, we have $\Delta u \not\equiv 0$. Therefore, there exists a point x and $r > 0$, such that $\Delta u > 0$ (or equivalently $\Delta u < 0$) in $\overline{B_r}(x) \subset \Omega$. The reasoning used in step 1 yields the following contradiction:

$$0 = \varphi_r'(x) = \frac{1}{\omega_n r^{n-1}} \int\limits_{B_r(x)} \Delta u(y) dy > 0.$$

□

Theorem 2.6 (Strong maximum principle). *For a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ that is harmonic in Ω we have*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Furthermore, if Ω is connected and there exists $x_0 \in \Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u$, then u is constant in Ω.

Proof. **Step 1.** Suppose that Ω is connected and that there exists $x_0 \in \Omega$, such that $u(x_0) = \max_{\bar{\Omega}} u =: M$, i.e. u attains its maximum over $\overline{\Omega}$ in $x_0 \in \Omega$.
We define a set $V = \{x \in \Omega \mid u(x) = M\}$. V is relatively closed in Ω because $V = u^{-1}[\{M\}]$, i.e. V is the preimage of the closed set $\{M\}$, making its preimage a closed set as well. On the other hand, since Ω is open and V non-empty, for any $v \in V$ we can find $r > 0$ such that $B_r(v) \subset \Omega$. A simple application of the mean value property immediately shows that $u \equiv M$ in $B_r(v)$.

$$M = u(v) = \frac{1}{|B_r(v)|} \int\limits_{B_r(v)} u(y) dy \leq \frac{1}{|B_r(v)|} \int\limits_{B_r(v)} M dy \leq M.$$

Therefore, we have $B_r(v) \subset V$, making V relatively open in Ω. This implies that $V = \Omega$, since by hypothesis Ω is connected, which means that only Ω and $\emptyset$ can be relatively clopen in Ω. Since $x_0 \in V$, V is non-empty and therefore we have $V = \Omega$. This proves the second claim of the theorem. Additionally, if Ω is connected the second claim of the theorem directly implies the first claim.

Step 2. What remains to be shown is that for Ω not connected the first claim of the theorem still holds. If the maximum of u is not attained in Ω, then the first claim of the theorem holds. If the maximum of u is attained in Ω, there exists a point $x \in \Omega$, such that $u(x) = \max_{\overline{\Omega}} u =: M$. We define the set V as in step 1. Additionally we define C as the intersection of all in Ω relatively clopen sets containing x. Let $\tilde{V} := V \cap C$. Since $\mathbb{R}^n$ is locally connected, Ω is locally connected as well. We know that in a locally connected space the intersection of clopen sets containing x is equal to the connected component of Ω containing x, i.e. the maximally connected subset of Ω containing x with inclusion as the criteria for the maximum. Therefore C is connected and the same reasoning we used in step 1, applied to $\tilde{V}$ and C, holds and shows that $\tilde{V} = C$.
By lemma A.15, we know that $\partial\tilde{V} \subset \partial\Omega$. Since the only clopen sets in $\mathbb{R}^n$ are the empty set and $\mathbb{R}^n$ we that there exists some point $y \in \partial\Omega \cap \partial\tilde{V}$. Since $u|_{\overline{\tilde{V}}} \equiv M$, we conclude that $u(y) = M$ and thus u attains its maximum on the boundary of Ω. □

2.2 Superharmonic and subharmonic functions

Definition 2.7. A function $u \in C^2(\Omega)$ is

i) subharmonic in Ω if $-\Delta u \leq 0$ in Ω,

ii) superharmonic in Ω if $-\Delta u \geq 0$ in Ω.

Theorem 2.8 (Mean value inequality). *Given a function $u \in C^2(\Omega)$, a point $x \in \Omega$ and $r > 0$ such that $B_r(x) \subset \Omega$ the following claims hold.*

i) If $-\Delta u > 0$ in $B_r(x)$, then for any $0 < t < r$ we have

$$u(x) > \frac{1}{|\partial B_t(x)|} \int\limits_{\partial B_t(x)} u(y) d\sigma_y.$$

ii) If $-\Delta u < 0$ in $B_r(x)$, then for any $0 < t < r$ we have

$$u(x) < \frac{1}{|\partial B_t(x)|} \int\limits_{\partial B_t(x)} u(y) d\sigma_y.$$

Proof. We start out by proving the first part of the theorem. The method used in theorem 2.5 can be used to show that for any $s \in [0, t]$, we have that if $-\Delta u > 0$, then

$$0 > \frac{1}{s^{n-1}} \int\limits_{B_s(x)} \Delta u(x) dx = \int\limits_{\partial B_1(0)} \frac{\partial u}{\partial s}(x + s\omega) d\sigma_\omega = \frac{\partial}{\partial s} \int\limits_{\partial B_1(0)} u(x + s\omega) d\sigma_\omega.$$

Integrating with respect to s from 0 to t and a subsequent trivial rearrangement yields

$$\frac{1}{|\partial B_1(0)|} \int\limits_{\partial B_1(0)} u(x+t\omega)d\sigma_\omega < u(x).$$

To show the claim for $-\Delta u < 0$ we simply switch the inequalities around. □

Lemma 2.9. *Under the assumption of theorem 2.8 the following claims hold.*

i) If x_o is a minimum point of u in Ω, then $-\Delta u(x_0) \leq 0$.

ii) If x_o is a maximum point of u in Ω, then $-\Delta u(x_0) \geq 0$.

Proof. We prove the first claim of the lemma. The second claim follows from the same line of reasoning.
Suppose x_0 is a minimum point of u in Ω with $-\Delta(x_0) > 0$. Since Δu is continuous in Ω, there exists a neighbourhood of x_0, which we denote by $B_\delta(x_0)$, in which $-\Delta u > 0$. We make use of theorem 2.8, which states that

$$u(x_0) > \frac{1}{|\partial B_{\delta'}(x_0)|} \int\limits_{\partial B_{\delta'}(x_0)} u(y)d\sigma_y,$$

for any $0 < \delta' < \delta$. This contradicts the assumption that x_0 is a local minimum of u. □

Theorem 2.10 (Weak maximum principle)**.** *For a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$, we have that*

i) $-\Delta u \geq 0$ in $\Omega \Rightarrow \min\limits_{\bar{\Omega}} u \geq \min\limits_{\partial\Omega} u$,

ii) $-\Delta u \leq 0$ in $\Omega \Rightarrow \max\limits_{\bar{\Omega}} u \leq \max\limits_{\partial\Omega} u$.

Proof. We will only show the first claim, since the second claim can be shown using the same argument. Suppose $-\Delta u \geq 0$. We define $u_\delta := u - \delta|x|^2$ for any $\delta > 0$. The function u_δ is strictly superharmonic, i.e. $-\Delta u_\delta = -\Delta u + 2\delta n > 0$. Suppose that x_0 is a minimum of u_δ in Ω. By lemma 2.9, we have $-\Delta u_\delta(x_0) \leq 0$, which is a contradiction. Therefore, $\min\limits_{\bar{\Omega}} u_\delta \geq \min\limits_{\partial\Omega} u_\delta$. Letting $\delta \to 0$, we arrive at the desired conclusion. □

Remark 2.11. We say that a function u is smooth if it is of C^∞ regularity.

Theorem 2.12 (Removable Discontinuity)**.** *For a function u and some $R > 0$ such that $u \in C(\partial B_R(0)) \cap C^2(B_R(0)\backslash\{0\})$ is harmonic in $B_R(0)\{0\}$ and satisfies the following condition,*

$$\begin{cases} \lim\limits_{|x|\to 0} \frac{u(x)}{\log|x|} = 0 & \text{if } n = 2, \\ \lim\limits_{|x|\to 0} \frac{u(x)}{|x|^{2-n}} = 0 & \text{if } n \geq 3, \end{cases}$$

we can redefine u, such that u smooth and harmonic in all of $B_R(0)$.

Proof. **Case 1.** $n = 2$.
Suppose that u is continuous in $B_R(0)\setminus\{0\}$, $\lim_{|x|\to 0}\frac{u(x)}{\log|x|} = 0$, and suppose that v is a solution of the problem

$$\begin{cases} \Delta v = 0 & \text{in } B_R(0), \\ v = u & \text{on } \partial B_R(0). \end{cases}$$

The existence of such a function $v \in C^\infty(B_R(0))$ is guaranteed by theorem 3.7. All we need to show is that $u \equiv v$ in $B_R(0)\setminus\{0\}$. Let

$$w := v - u \text{ in } B_R(0)\setminus\{0\} \quad \text{and} \quad M_r := \max_{\partial B_r(0)} |w| \text{ for any } r > 0, \text{ such that } r < R.$$

For any $x \in \partial B_r(0)$, we have

$$-M_r \frac{\log\frac{|x|}{R}}{\log\frac{r}{R}} \le w(x) \le M_r \frac{\log\frac{|x|}{R}}{\log\frac{r}{R}}.$$

Since $\log|x|$ and $w(x)$ are harmonic functions this holds not only on $\partial B_r(0)$ but also in $B_r(0)$ thanks to the weak maximum principle 2.10. Thus,

$$|w(x)| \le M_r \frac{\log\frac{|x|}{R}}{\log\frac{r}{R}},$$

for any $x \in B_R(0)\setminus B_r(0)$. We can establish a boundary for M_r. Namely, we get

$$\begin{aligned} M_r &= \max_{\partial B_r(0)} |v - u| \le \max_{\partial B_r(0)} |v| + \max_{\partial B_r(0)} |u| \le \max_{\partial B_R(0)} |v| + \max_{\partial B_r(0)} |u| \\ &\le \max_{\partial B_R(0)} |u| + \max_{\partial B_r(0)} |u|. \end{aligned}$$

Therefore, for any fixed x such that $0 < |x| < R$, we have

$$|w(x)| \le \max_{\partial B_R(0)} |u| \frac{\log\frac{|x|}{R}}{\log\frac{r}{R}} + \max_{\partial B_r(0)} |u| \frac{\log\frac{|x|}{R}}{\log\frac{r}{R}},$$

with any r such that $0 < r < |x|$. If $r \to 0$ we conclude that $|w(x)| \to 0$. This follows from the assumption that $u = o(\log|x|)$, which implies that $\max_{\partial B_r(0)} u \frac{1}{\log(r)} \to 0$. Thus, we arrive at the desired result, i.e. $w \equiv 0$ in $B_R(0)\setminus\{0\}$.

Case 2. $n \ge 3$.
We use the same method as for the case $n = 2$ to establish the following boundary:

$$|w(x)| \le M_r \frac{x^{2-n}}{r^{2-n}},$$

for any $x \in B_R(0)\setminus B_r(0)$. Using the same boundary as before for M_r, we show that $|w(x)| \to 0$ as $r \to 0$. □

2.3 Additional properties of harmonic functions

Theorem 2.13 (C^∞ regularity of harmonic functions)**.** *If $u \in C(\Omega)$ satisfies the mean value property in Ω, then $u \in C^\infty(\Omega)$.*

Proof. Let η be the standard mollifier and let $u_\epsilon := \eta_\epsilon * u$ in Ω_ϵ, where $\eta_\epsilon(x) := \frac{1}{\epsilon}\eta(\frac{x}{\epsilon})$ and $\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}$. We know that $\eta \in C_c^\infty(\mathbb{R}^n)$ and $u_\epsilon \in C^\infty(\Omega_\epsilon)$. For more details and an in-depth proof of this claim, the reader may consult [8]. Our aim is to show that $u_\epsilon(x) = u(x)$. We will do so by making use of the mean-value property.

$$\begin{aligned} u_\epsilon(x) &= \int\limits_{\Omega} \eta_\epsilon(x-y)u(y)dy = \frac{1}{\epsilon^n} \int\limits_{B_\epsilon(x)} \eta\Big(\frac{x-y}{\epsilon}\Big)u(y)dy \\ &= \frac{1}{\epsilon^n}\int\limits_0^\epsilon \eta\big(\frac{r}{\epsilon}\big)\Big(\int\limits_{\partial B_r(x)} u(y)d\sigma_y\Big)dr = \frac{1}{\epsilon^n}\int\limits_0^\epsilon \eta\big(\frac{r}{\epsilon}\big)\frac{\omega_n}{n}r^{n-1}u(x)dr \\ &= u(x)\int\limits_{B_\epsilon(0)} \eta_\epsilon(y)dy = u(x). \end{aligned} \tag{2.2}$$

This concludes the proof, since $u \equiv u_\epsilon$ in Ω_ϵ, which means that $u \in C^\infty(\Omega_\epsilon)$ for every $\epsilon > 0$. ☐

Theorem 2.14 (Pointwise estimates for derivatives)**.** *Let u be a harmonic function in Ω. Then for every $B_r(x) \subset \Omega$ and a multi-index α, the following pointwise estimates for the derivatives of u hold.*

$$|D^\alpha u(x)| \leq \frac{C_{|\alpha|}}{r^{n+|\alpha|}}\|u\|_{L^1(B_r(x))}, \tag{2.3}$$

where $C_0 = \frac{n}{\omega_n}$, $C_i = \frac{(2^{n+1}i)^i n^{i+1}}{\omega_n}$ for $i = 1, 2, ...$

Proof. We prove this theorem via induction.

For $i = 0$ this is trivial, since equation (2.3) is equivalent to the mean value property that holds for all harmonic functions.

For $i = 1$ we simply recall that for any $j = 1...n$ u_{x_j} is harmonic due to u being harmonic. Therefore, u_{x_j} fulfills the mean value property and thus we have

$$\begin{aligned} |u_{x_j}(x)| &= \Big|\tfrac{1}{|B_{r/2}(x)|}\int\limits_{B_{r/2}(x)} u_{x_j}(y)dy\Big| = \Big|\tfrac{2^n n}{\omega_n r^n}\int\limits_{\partial B_{r/2}(x)} u(y)\nu_j d\sigma_y\Big| \\ &\leq \tfrac{2n}{r}\|u\|_{L^\infty(\partial B_{r/2}(x))}, \end{aligned} \tag{2.4}$$

where the second equality follows a trivial application of the Gauss divergence theorem A.9. For any $y \in \partial B_{r/2}(x)$, we have $B_{r/2}(x) \subset B_r(x) \subset \Omega$ and therefore

$$|u(y)| \leq \frac{2^n n}{\omega_n r^n}\|u\|_{L^1(B_r(x))}, \tag{2.5}$$

from the previously established result for $i = 0$. Inserting equation (2.5) into equation (2.4), namely replacing $\|u\|_{L^\infty}$, completes the proof for the case $i = 1$.

Suppose that $i \geq 2$ and that the estimates of the theorem hold for each multi-index β of order smaller than i. We fix $B_r(x) \in \Omega$ and take a multi-index α of order i. We can express $D^\alpha u$ via $D^\beta u$, namely for some $j \in \{1, ..., n\}$, we have that $D^\alpha u = (D^\beta u)_{x_j}$. Since $D^\beta u$ is harmonic, we can use the same reasoning as above, which yields

$$|D^\alpha u(x)| \leq \frac{ni}{r} \|D^\beta u\|_{L^\infty(\partial B_{r/i}(x))}. \tag{2.6}$$

For each $y \in B_{r/i}(x)$ we have $B_{r(i-1)/i}(y) \subset B_r(x) \subset \Omega$ and hence we may use the estimates of the theorem as follows:

$$|D^\beta u(y)| \leq \frac{(2^{n+1}(i-1))^{i-1} n^i}{\omega_n (\frac{i-1}{i} r)^{n+i-1}} \|u\|_{L^1(B_r(x))}. \tag{2.7}$$

□

Theorem 2.15 (Liouville)**.** *A harmonic and bounded function $u : \mathbb{R}^n \to \mathbb{R}$ is constant.*

Proof. We take $r > 0$ and fix $x \in \mathbb{R}^n$. Then we apply theorem 2.14 on $B_r(x)$. Hence we get

$$|Du(x)| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B_r(x))} \leq \frac{C_1 \omega_n r^n}{r^{n+1} n} \|u\|_{L^\infty(B_r(x))} \leq \frac{C}{r} \to 0,$$

as $r \to \infty$. Thus, u is constant. □

Theorem 2.16 (Harnack's inequality)**.** *For every connected open set $V \subset\subset \Omega$, there exists a constant C that only depends on V such that for any non-negative harmonic function u in Ω the following holds*

$$\sup_V u \leq C \inf_V u.$$

Proof. Let $r := \frac{1}{4}\text{dist}(V, \partial\Omega)$. We choose $x, y \in V$ such that $|x - y| \leq r$. This implies that

$$\begin{aligned} u(x) &= \frac{1}{|B_{2r}(x)|} \int_{B_{2r}(x)} u(z)dz \geq \frac{n}{\omega_n 2^n r^n} \int_{B_r(y)} u(z)dz \\ &= \frac{1}{2^n |B_r(y)|} \int_{B_r(y)} u(z)dz = \frac{1}{2^n} u(y). \end{aligned} \tag{2.8}$$

Using this inequality twice we find that $\frac{1}{2^n} u(y) \leq u(x) \leq 2^n u(y)$ for all $x, y \in V$ such that $|x - y| \leq r$. Since the closure of V is compact there exists a finite family of balls $\{B_i\}_{i=1}^N$, all of radius $\frac{r}{2}$, that cover $\bar{V}$. On top of that by hypothesis V is connected, which allows us to create not only a finite covering of V but do so with a chain of balls, meaning that the family of balls $\{B_i\}_{i=1}^N$ is ordered such that $B_i \cap B_{i-1} \neq \emptyset$. This completes the proof, since for any $x, y \in V$, we also have $x \in B_k$, $y \in B_j$ for $k, j \in \{1, ..., N\}$, which means that

$$u(x) \geq \left(\frac{1}{2^n}\right)^{|j-k|+1} u(y) \geq \frac{1}{2^{n(N+1)}} u(y).$$

□

3 Green's function

In this section, we will develop the fundamental solution of the Laplace equation and solve the Poisson equation over $\mathbb{R}^n$ and for any ball. This section is based on the material covered during Professor Radulescu's lectures [7]. Additionally, this chapter is based on results that can be found in [2].

3.1 Fundamental solution of the Laplace equation

The function

$$\Gamma(x) := \begin{cases} \frac{1}{2\pi}\log|x| & \text{if } n = 2, \\ \frac{1}{\omega_n(n-2)}\frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \end{cases} \tag{3.1}$$

defined over $\mathbb{R}^n \backslash \{0\}$, is the fundamental solution of the Laplace equation.

Definition 3.1. We define the Newtonian potential v of $f \in L^p(\Omega)$ for $1 < p < \infty$ as follows.

$$v(x) := \int\limits_{\mathbb{R}^n} \Gamma(x-y)f(y)dy.$$

Finding the fundamental solution of the Laplace equation is a relatively trivial matter. We start out by trying to find a function $w : \mathbb{R} \to \mathbb{R}$ such that $u(x) = w(|x|)$ is a solution to the Laplace equation, where $|x| = \big(\sum\limits_{i=1}^{n} x_i^2\big)^{\frac{1}{2}}$. For any $i = 1, ..., n$, we have

$$\frac{\partial|x|}{\partial x_i} = \frac{1}{2|x|}\,2x_i = \frac{x_i}{|x|},$$

which gives us

$$\frac{\partial u}{\partial x_i} = w'(|x|)\frac{x_i}{|x|} \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i^2} = w''(|x|)\frac{x_i^2}{|x|^2} + w'(|x|)\big(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\big).$$

Thus we have $\Delta u = 0$ if and only if

$$\Delta u = w''(|x|)\sum_{i=1} n\tfrac{x_i^2}{|x|^2} + w'(|x|)\big(\tfrac{n}{|x|} - \sum_{i=1} n\tfrac{x_i^2}{|x|^3}\big) = w''(|x|) + \tfrac{n-1}{|x|}w'(|x|) = 0.$$

If $w' \neq 0$, we have $\frac{w''}{w'} = \frac{1-n}{|x|}$ and by integration we have $w'(|x|) = \frac{a}{|x|^{n-1}}$, where a is a constant. Therefore, we find that the function $w(|x|)$ takes the following form.

$$w(|x|) = \begin{cases} b\log(|x|) + c & \text{if } n = 2 \\ \frac{b}{|x|^{n-2}} + c & \text{if } n \geq 3, \end{cases}$$

where b, c are constants.

Theorem 3.2 (Fundamental solution of the Poisson equation)**.** *Let u be the Newtonian potential of a function $f \in C^2_c(\mathbb{R}^n)$. Then* *i)* $u \in C^2(\mathbb{R}^n)$,

ii) $-\Delta u = f$ *in* $\mathbb{R}^n$.

Proof. **Step 1.** Clearly,

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)dy = \int_{\mathbb{R}^n} \Gamma(y)f(x-y)dy,$$

therefore,

$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \Gamma(y)\Big(\frac{f(x+he_i-y)-f(x-y)}{h}\Big)dy,$$

where $h \neq 0$ and $e_i = (0, ..., 1, 0, ..., 0)$ where 1 is in the i-th slot. Of course,

$$\frac{f(x+he_i-y)-f(x-y)}{h} \to f_{x_i}(x-y) \text{ uniformly on } \mathbb{R}^n \text{ as } h \to 0,$$

and thus for $i = 1, 2, ..., n$,

$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Gamma(y)f_{x_i}(x-y)dy.$$

Likewise for $i = 1, 2, ..., n$,

$$u_{x_ix_j}(x) = \int_{\mathbb{R}^n} \Gamma(y)f_{x_ix_j}(x-y)dy$$

and thus, $u \in C^2(\mathbb{R}^n)$, since the right-hand side of the last identity is continuous.

Step 2. Fix $\varepsilon > 0$. Due to the singularity of the fundamental solution at the origin, we have to be careful in our calculation. Namely, we first consider the splitting

$$\begin{aligned}\Delta u(x) &= \int_{B_\varepsilon(0)} \Gamma(y)\Delta_x f(x-y)dy + \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Gamma(y)\Delta_x f(x-y)dy \\ &=: I_\varepsilon^1 + I_\varepsilon^2.\end{aligned} \tag{3.2}$$

Then, polar coordinates implies

$$|I_\varepsilon^1| \leq C\|D^2 f\|_{L^\infty(\mathbb{R}^n)}|\Gamma(y)|dy \leq C\varepsilon^{n-(n-2)} \leq C\varepsilon^2. \tag{3.3}$$

Integration by parts implies

$$\begin{aligned}I_\varepsilon^2 &= \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Gamma(y)\Delta_y f(x-y)dy \\ &= \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} D\Gamma(y)\cdot D_y f(x-y)dy + \int_{\partial B_\varepsilon(0)} \Gamma(y)\frac{\partial f}{\partial \nu}(x-y)d\sigma_y \\ &=: J_\varepsilon^1 + J^2\varepsilon,\end{aligned} \tag{3.4}$$

where ν denotes the inward pointing unit normal along $\partial B_\varepsilon(0)$. Now,

$$|J^2_\varepsilon| \leq \|Df\|_{L^\infty(\mathbb{R}^n)}|\Gamma(y)|d\sigma_y \leq C\varepsilon. \tag{3.5}$$

Another integration by parts and using the harmonic property of Γ yields

$$\begin{aligned} J^1\varepsilon &= \int\limits_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Delta\Gamma(y)f(x-y)dy - \int\limits_{\partial B_\varepsilon(0)} \frac{\partial\Gamma}{\partial\nu}(y)f(x-y)d\sigma_y \\ &= -\int\limits_{\partial B_\varepsilon(0)} \frac{\partial\Gamma}{\partial\nu}(y)f(x-y)d\sigma_y. \end{aligned} \tag{3.6}$$

It is clear, that $D\Gamma(y) = -\frac{y}{\omega_n|y|^n}$ (with $y \neq 0$) and $\nu = -\frac{y}{|y|} = -\frac{y}{\varepsilon}$ on $\partial B_\varepsilon(0)$. Thus,

$$\frac{\partial\Gamma}{\partial\nu}(y) = \nu \cdot D\Gamma(y) = \frac{1}{\omega_n\varepsilon^{n-1}} \text{ on } \partial B_\varepsilon(0).$$

Therefore,

$$\begin{aligned} J^1_\varepsilon &= -\frac{1}{\omega_n\varepsilon^{n-1}} \int\limits_{\partial B_\varepsilon(0)} f(x-y)d\sigma_y \\ &= -\frac{1}{|B_\varepsilon(0)|} \int\limits_{\partial B_\varepsilon(0)} f(x-y)d\sigma_y \to -f(x), \end{aligned} \tag{3.7}$$

as $\varepsilon \to 0$. Hence, by combining the equations (3.3), (3.4), (3.5), (3.6), (3.7) and sending $\varepsilon \to 0$ in equation (3.2), we obtain the desired result $-\Delta u(x) = f(x)$.

For $n = 2$, we change the estimates for I^1_ε and $J^2\varepsilon$ as follows:

$$|I^1\varepsilon| \leq C\varepsilon^2|\log\varepsilon| \quad \text{and} \quad |J^2_\varepsilon| \leq C\varepsilon|\log\varepsilon|.$$

With those changes in place, the proof remains valid for $n = 2$. □

3.2 Representation of solutions

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with C^1 boundary. A set has a C^1 boundary if for each $x_0 \in \partial\Omega$, there exists $r > 0$ and a function $\gamma \in C^{\mathbb{R}^n}$ such that

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) \mid x_n > \gamma(x_1, x_2, ..., x_{n-1})\},$$

where if needed we reorient the coordinate axes. Our goal is to find a representation of the solution of the Poisson equation

$$-\Delta u = f \text{ in } \Omega,$$

subject to the boundary condition

$$u = g \text{ on } \partial\Omega.$$

Fix $x \in \Omega$, let $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \Omega$. We use Green's formulas A.7 on $V_\varepsilon = \Omega \backslash B_\varepsilon(x)$ to show that

$$\begin{aligned} &\int\limits_{V_\varepsilon} u(y)\Delta\Gamma(y-x) - \Gamma(y-x)\Delta u(y)dy \\ &= \int\limits_{\partial V_\varepsilon} u(y)\frac{\partial \Gamma}{\partial \nu}(y-x) - \Gamma(y-x)\frac{\partial u}{\partial \nu}(y)d\sigma_y. \end{aligned} \tag{3.8}$$

Since $\Delta\Gamma(x-y) = 0$ for $x \neq y$ and

$$\Big| \int\limits_{\partial B_\varepsilon(x)} \Gamma(y-x)\frac{\partial u}{\partial \nu}(y)d\sigma_y \Big| \leq C\varepsilon^{n-1} \max_{\partial B_\varepsilon(0)} |\Gamma| = o(1),$$

we can show (as was done in theorem 3.2) that

$$\int\limits_{\partial B_\varepsilon(x)} u(y)\frac{\partial \Gamma}{\partial \nu}(y-x)d\sigma_y = \frac{1}{|\partial B_\varepsilon(x)|} \int_{\partial B_\varepsilon(x)} u(y)d\sigma_y \to u(x), \tag{3.9}$$

as $\varepsilon \to 0$.
Using equation (3.8) and equation (3.9), we show that

$$\begin{aligned} u(x) = &\int\limits_{\partial\Omega} \big(\Gamma(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Gamma}{\partial \nu}(y-x)\big)d\sigma_y \\ &- \int\limits_{\Omega} \Gamma(y-x)\Delta u(y)dy. \end{aligned} \tag{3.10}$$

The equation (3.10) holds for all $x \in \Omega$ and all $u \in C^2(\Omega)$. We know that u satisfies the Poisson equation and the boundary values of u on $\partial\Omega$ are known. The only unknown term in (3.10) is $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$. To address this we introduce a function $\phi_x(y)$ for any fixed $x \in \Omega$, such that ϕ_x solves the boundary-value problem

$$\begin{cases} \Delta\phi_x = 0 & \text{in } \Omega, \\ \phi_x = \Gamma(y-x) & \text{on } \partial\Omega. \end{cases}$$

We apply Green's formula once more to obtain

$$\begin{aligned} -\int\limits_{\Omega} \phi_x(y)\Delta u(y)dy &= \int\limits_{\partial\Omega} u(y)\frac{\partial \phi_x}{\partial \nu}(y) - \phi_x(y)\frac{\partial u}{\partial \nu}(y)d\sigma_y \\ &= \int\limits_{\partial\Omega} u(y)\frac{\partial \phi_x}{\partial \nu}(y) - \Gamma(y-x)\frac{\partial u}{\partial \nu}(y)d\sigma_y. \end{aligned} \tag{3.11}$$

Definition 3.3 (Green's function for Ω)**.** We introduce the Green function for Ω.

$$G(x,y) := \Gamma(y-x) - \phi_x(y) \quad \text{for} \quad x, y \in \Omega, x \neq y.$$

Using this definition allows us to get rid of any terms that include $\frac{\partial u}{\partial \nu}$. We add equation (3.11) to equation (3.10) and arrive at

$$u(x) = -\int\limits_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y)d\sigma_y - \int\limits_{\Omega} G(x,y)\Delta u(y)dy,$$

where $\frac{\partial G}{\partial \nu}(x,y)$ is the outer normal derivative of G with respect to y.

To summarize, if we have a function $u \in C^2(\bar{\Omega})$ that is a solution of the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = g & \text{on } \partial\Omega, \end{cases}$$

for two given continuous functions f and g, then the following holds:

$$u(x) = -\int\limits_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)d\sigma_y + \int\limits_{\Omega} G(x,y)f(y)dy. \tag{3.12}$$

Remark 3.4. For simple shapes Ω we can compute the corrector function ϕ_x.

3.3 Green's function for the unit ball

We now set out to explicitly compute the corrector function ϕ_x for the unit ball. For any $x \in \mathbb{R}^n \backslash \{0\}$, we define $\tilde{x} := \frac{x}{|x|^2}$. We choose a fixed $x \in B_1(0)$ and we introduce the following problem

$$\begin{cases} \Delta\phi_x = 0 & \text{in } B_1(0), \\ \quad \phi_x = \Gamma(y-x) & \text{on } \partial B_1(0), \end{cases}$$

where $G(x,y) = \Gamma(y-x) - \phi_x(y)$ is Green's function.

Suppose that $n \geq 3$. $y \mapsto \Gamma(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$ and thus $y \mapsto |x|^{2-n}\Gamma(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$ as well. We can therefore say that $\phi_x(y) := \Gamma(|x|(y-\tilde{x}))$ is harmonic in $B_1(0)$. Furthermore, for $y \in \partial B_1(0)$ and $x \neq 0$, we have

$$|x|^2|y-\tilde{x}|^2 = |x|^2\big(|y|^2 - 2\frac{y\cdot x}{|x|} + \frac{1}{|x|^2}\big) = |x|^2 - 2y\cdot x + 1 = |x-y|^2.$$

Therefore, $|x-y|^{2-n} = (|x|\ |y-\tilde{x}|)^{2-n}$ and we end up with

$$\phi_x(y) = \Gamma(y-x) \text{ for } y \in \partial B_1(0),$$

as required.

If $n = 2$, we can apply the same procedure and we end up with the same result. This allows Green's function to be defined universally, independent of whether $n \geq 3$ or $n = 2$.

$$G(x,y) := \Gamma(y-x) - \Gamma(|x|(y-\tilde{x})), \tag{3.13}$$

for $x, y \in B_1(0)$.

Suppose the function u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B_1(0), \\ u = g & \text{on } \partial B_1(0), \end{cases}$$

Taking a look at the representation formula equation (3.12) reveals that

$$u(x) = -\int\limits_{\partial B_1(0)} g(y)\frac{\partial G}{\partial \nu}(x,y)d\sigma_y. \tag{3.14}$$

According to equation (3.13),

$$\frac{\partial G}{\partial y_i}(x,y) = \frac{\partial \Gamma}{\partial y_i}(y-x) - \frac{\partial \Gamma(|x|(y-\tilde{x}))}{\partial y_i}.$$

By simple calculation, we get

$$\frac{\partial \Gamma}{\partial y_i}(y-x) = \frac{1}{\omega_n}\frac{x_i - y_i}{|x-y|^n} \quad \text{and} \quad \frac{\partial \Gamma(|x|(y-\tilde{x}))}{\partial y_i} = -\frac{1}{\omega_n}\frac{y_i|x|^2 - x_i}{|x-y|^n},$$

for any $y \in \partial B_1(0)$. Therefore,

$$\begin{aligned} \frac{\partial G}{\partial \nu}(x,y) &= \sum_{i=1}^{n} y_i \frac{\partial G}{\partial y_i}(x,y) \\ &= -\frac{1}{\omega_n}\frac{1}{|x-y|^n}\sum_{i=1}^{n} y_i((y_i - x_i) - y_i|x|^2 + x_i) \\ &= -\frac{1}{\omega_n}\frac{1-|x|^2}{|x-y|^n}. \end{aligned} \tag{3.15}$$

The combination of equation (3.15) and equation (3.13) gives us the representation formula

$$u(x) = \frac{1-|x|^2}{\omega_n}\int\limits_{\partial B_1(0)} \frac{g(y)}{|x-y|^n}d\sigma_y.$$

We can extend the representation formula to any ball of arbitrary radius using dilation.

Definition 3.5 (Poisson's formula)**.**

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R}\int\limits_{\partial B_R(0)} \frac{g(y)}{|x-y|^n}d\sigma_y,$$

where $x \in B_R(0)$.

Definition 3.6 (Poisson's kernel)**.**

$$K(x,y) := \frac{R^2 - |x|^2}{\omega_n R}\frac{1}{|x-y|^n},$$

where $x \in B_R(0)$, $y \in \partial B_R(0)$.

All of the above culminates in the following result.

Theorem 3.7 (Poisson's formula for balls)**.** *Let g be a continuous function on $\partial B_R(0)$. We define the function u using the Poisson formula 3.5. Then*

i) $u \in C^\infty(B_R(0))$,

ii) $\Delta u = 0$ *in* $B_R(0)$,

iii) $\lim\limits_{B_R(0) \ni x \to x_0} u(x) = g(x_0)$ *for each* $x_0 \in \partial B_R(0)$.

4 Existence theory

This chapter contains two major existence results and provides an introduction to sobolev spaces, the notion of weak solutions and elliptic operators.

4.1 Perron method

In this section we will establish an existence and uniqueness result for classical solutions to Dirichlet problems on general domains. The approach in this section loosely follows [3]. Apart from following the established approach to prove the perron method, the author developed the content of this chapter independently.

The perron method relies on the existence of solutions on ball domains. While we will only consider the case of the Laplacian operator, the perron method can be extended to more general elliptic operators. We consider the following problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and φ is a continuous function on $\partial\Omega$. Additionally we require that Ω satisfies the exterior sphere condition, which we will discuss in more detail later.

If Ω is an open ball, then the solutions of equation (4.1) are given by the Poisson formula and the Green function for ball domains. The purpose of the Perron method is to prove the existence of a unique solution if Ω is not a ball domain.

We start out by providing an alternative definition of subharmonic and superharmonic continuous functions based on the maximum principle. This is in contrast to defining subharmonic and superharmonic function using the the more traditional definition that involves the laplacian.

Definition 4.1. Let Ω be a bounded domain in $\mathbb{R}^n$ and v be a continuous function in Ω. Then v is subharmonic (respectively superharmonic) in Ω if for any ball $B \subset \Omega$ and any harmonic function $w \in C(\bar{B})$,

$$\left\{ \begin{array}{l} v \le w \\ v \ge w \end{array} \right. \text{on } \partial B \Rightarrow \left\{ \begin{array}{l} v \le w \\ v \ge w \end{array} \right. \text{in } B. \quad \begin{array}{r} \text{(Subharmonic case)} \\ \text{(Superharmonic case)} \end{array}$$

Remark 4.2. If $v \in C^2(\Omega)$ is subharmonic in Ω as defined in definition 4.1, then it is also subharmonic in the traditional sense given in definition 2.7.

Lemma 4.3. *Let $u, v \in C(\bar{\Omega})$, where Ω is a bounded open subset of $\mathbb{R}^n$. If u is subharmonic in Ω, v is superharmonic in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

Proof. **Step 1.** We can assume, without loss of generality, that Ω is connected. To justify this assumption, let us assume that Ω is not connected. We denote the family of connected components of Ω by C. Thus, every $V \in C$ is a maximally connected subset of Ω, and C forms a partition of Ω.

We know that $u \leq v$ on $\partial\Omega$. Using lemma A.15, we know that $\partial V \subset \partial\Omega$ for any $V \in C$. Therefore, $u \leq v$ on ∂V for any $V \in C$. Since V is connected, if the lemma holds for connected sets, we can conclude that $u \leq v$ in V. Since this is true for any $V \in C$ and C forms a partition of Ω, we have $u \leq v$ in Ω. Therefore, we can show that the lemma holds for a connected bounded domain Ω, without loss of generality.

Step 2. We show that the lemma holds if Ω is connected. Let $M := \max_{\bar{\Omega}}(u-v)$ and $D := \{x \in \Omega | u(x) - v(x) = M\} \subset \Omega$. D is relatively clopen in Ω. Since Ω is connected, D can only be equal to the empty set or to Ω. To show that D is indeed relatively clopen in Ω we remark that the continuity of $u - v$ immediately proves relative closedness. To show that D is open, we take any point $x_0 \in D$ and $0 < r < \text{dist}(x_0, \partial\Omega)$. We define two problems.

$$(1) \begin{cases} \Delta\bar{u} = 0 & \text{in } B_r(x_0), \\ \bar{u} = u & \text{on } \partial B_r(x_0) \end{cases} \quad (2) \begin{cases} \Delta\bar{v} = 0 & \text{in } B_r(x_0), \\ \bar{v} = v & \text{on } \partial B_r(x_0). \end{cases}$$

The existence of a solution for each of these problems is a consequence of the Poisson formula for $\Omega = B_r(x_0)$. We denote these solutions by $\bar{u}$ and $\bar{v}$. Moreover, we have $u \leq \bar{u}$ and $\bar{v} \leq v$ in $B_r(x_0)$. Therefore,

$$\bar{u} - \bar{v} \geq u - v \text{ in } B_r(x_0).$$

Thus,

$$\begin{cases} \Delta(\bar{u} - \bar{v}) = 0 & \text{in } B_r(x_0), \\ \bar{u} - \bar{v} = u - v & \text{on } \partial B_r(x_0). \end{cases}$$

With $u - v \leq M$ on $\partial B_r(x_0)$, by the maximum principle, we have $\bar{u} - \bar{v} \leq M$ in $B_r(x_0)$. In particular,

$$M \geq (\bar{u} - \bar{v})(x_0) \geq (u - v)(x_0) = M.$$

Hence, $(\bar{u} - \bar{v})(x_0) = M$, which implies that $\bar{u} - \bar{v}$ has an interior maximum at x_0. By the strong maximum principle, $\bar{u} - \bar{v} \equiv M$ in $B_r(x_0)$ and thus $B_r(x_0) \subset D$ for all $0 < r < \text{dist}(x_0, \partial\Omega)$. We conclude that $D = \emptyset$ or $D = \Omega$. In other words either $u - v$ attains its maximum exclusively on $\partial\Omega$ or $u - v$ is constant in Ω. If $u - v$ is constant in Ω we can directly extend the hypothesis $u \leq v$ on $\partial\Omega$ to Ω. If $u - v$ attains its maximum exclusively on $\partial\Omega$ we conclude in a similar manner that $\max_{\Omega} u - v < \max_{\partial\Omega} u - v \leq 0$. ☐

Lemma 4.4 (Subharmonic property of a harmonic lifting)**.** *Let $v \in C(\bar{\Omega})$ be a subharmonic function in Ω and $B \subset\subset \Omega$ a ball. Let $w = v$ in $\bar{\Omega} \backslash B$ and $\Delta w = 0$ in B. Then w is a subharmonic in Ω and $v \leq w$ in $\bar{\Omega}$.*

Proof. The existence of such a function w is guaranteed by the Poisson formula for $\Omega = B$. w is smooth in B and continuous in $\bar{\Omega}$. We also have $v \leq w$ in Ω by the definition of subharmonic functions . We take any $B' \subset\subset \Omega$ and harmonic function $u \in C(\bar{B}')$ with $w \leq u$ on $\partial B'$. By $v \leq w$ on $\partial B'$, we have $v \leq u$ on $\partial B'$. v is subharmonic and u is harmonic in B' with $v \leq u$ on $\partial B'$. By lemma 4.3, $v \leq u$ in B'. Therefore, $w \leq u$ in $B \backslash B'$. Both w and u are harmonic in $B \cap B'$ and $w \leq u$ on $\partial(B \cap B')$. By the maximum principle, we have $w \leq u$ in $B \cap B'$. Thus, $w \leq u$ in B'. We conclude that, by definition, w is subharmonic in Ω. □

We will attempt to solve equation (4.1). We define

$$u_\varphi(x) = \sup\{v(x) \mid v \in C(\bar{\Omega}) \text{ is subharmonic in } \Omega,\ v \leq \varphi \text{ on } \partial\Omega\}. \tag{4.2}$$

Our goal is to show that the function u_φ is a solution of the Dirichlet problem (4.1). We show that u_φ, as defined in equation (4.2), is harmonic in Ω. Let

$$S = \{v \in C(\bar{\Omega}) \mid v \text{ is subharmonic in } \Omega, v \leq \varphi \text{ on } \partial\Omega\}. \tag{4.3}$$

For any $x \in \Omega$,

$$u_\varphi(x) = \sup\{v(x) | v \in S\}.$$

Step 1. We show that u_φ is well defined. Let $m = \min_{\partial\Omega}\varphi$ and $M = \max_{\partial\Omega}\varphi$. We clearly have $m \in S$ and therefore S is non-empty. On the other hand, M is a constant function and clearly harmonic in Ω with $\varphi \leq M$ on $\partial\Omega$. By lemma 4.3, for any $v \in S$,

$$v \leq M \text{ in } \bar{\Omega},$$

and thus u_φ is well defined and we have $u_\varphi \leq M$ in Ω.

Step 2. We will show that S is closed for the maximum of a finite number of functions. Let $v_1, v_2, ..., v_k \in S$ be any finite number of elements of S and define $v := \max\{v_1, v_2, ..., v_k\}$. v is subharmonic in Ω and therefore $v \in S$.

Step 3. We prove that u_φ is harmonic in any ball $B_r(x_0) \subset \Omega$. By definition of u_φ, the existence of a sequence of functions $v_i \in S$ such that

$$\lim_{i \to \infty} v_i(x_0) = u_\varphi(x_0).$$

is guaranteed. We may now replace v_i in the above by any $\tilde{v}_i \in S$ with $\tilde{v}_i \geq v_i$ since

$$v_i(x_0) \leq \tilde{v}_i(x_0) \leq u_\varphi(x_0).$$

If necessary, we can replace v_i by $\max\{m, v_i\} \in S$. Therefore, we can assume that

$$m \leq v_i \leq u_\varphi \text{ in } \Omega. \tag{4.4}$$

For a fixed $B_r(x_0)$ and every v_i, we define w_i according to the definition of a harmonic lifting in lemma 4.4. Then $w_i = v_i$ in $\Omega \backslash B_r(x_0)$ and

$$\begin{cases} \Delta w_i = 0 \text{ in } B_r(x_0) \\ w_i = v_i \text{ on } \partial B_r(x_0). \end{cases}$$

By lemma 4.4, $w_i \in S$ and $v_i \leq w_i$ in Ω. Furthermore, w_i is harmonic in $B_r(x_0)$ and satisfies

$$\lim_{i\to\infty} w_i(x_0) = u_\varphi(x_0) \quad \text{and} \quad m \leq w_i \leq u_\varphi \text{ in } \Omega \quad \text{for any } i = 1, 2, ...$$

Since any bounded sequence of continuous functions on a compact set converges uniformly towards a continuous function, we know that there exists a continuous function w towards which a subsequence of $\{w_i\}$ converges uniformly. We will replace $\{w_i\}$ with this convergent subsequence for the sake of convenience. Thanks to the mean value property being satisfied by all functions w_i because they are all harmonic, we can extend this to the function w. Let $y \in B_r(x_0)$. Then for any s small enough such that $\overline{B_s(y)} \subset B_r(x_0)$ we have

$$w_i(y) = \frac{1}{\omega_n s^{n-1}} \int\limits_{\partial B_s(y)} w_i(x) d\sigma_x.$$

Thanks to uniform continuity on $\partial B_s(y)$ of w_i we can take the limit on both sides and move the limit on the right-hand side inside the integral, which yields

$$w(y) = \frac{1}{\omega_n s^{n-1}} \int\limits_{\partial B_s(y)} w(x) d\sigma_x.$$

Thus, w is harmonic in $B_r(x_0)$ and we have shown that

$$w \leq u_\varphi \text{ in } B_r(x_0) \quad \text{and} \quad w(x_0) = u_\varphi(x_0).$$

We now claim that $u_\varphi = w$ in $B_r(x_0)$. To show this, take any $\bar{x} \in B_r(x_0)$ and proceed as before, by replacing $\bar{x}$ with x_0. By definition of u_φ, there exists a sequence of $\{\bar{v}_i\} \subset S$ such that

$$\lim_{i\to\infty} \bar{v}_i(\bar{x}) = u_\varphi(\bar{x}).$$

As before, we can replace, if necessary, $\bar{v}_i$ by $\max\{\bar{v}_i, w_i\} \in S$. So we may also assume that

$$w_i \leq \bar{v}_i \leq u_\varphi \quad \text{in } \Omega.$$

For a fixed $B_r(x_0)$ and each $\bar{v}_i$, we let $\bar{w}_i$ be the harmonic lifting in lemma 4.4. Then $\bar{w}_i \in S$ and $\bar{v}_i \leq \bar{w}_i$ in Ω. Moreover, $\bar{w}_i$ is harmonic in $B_r(x_0)$ and satisfies

$$\lim_{i\to\infty} \bar{w}_i(\bar{x}) = u_\varphi(\bar{x}) \quad \text{and} \quad m \leq \max\{\bar{v}_i, w_i\} \leq \bar{w}_i \leq u_\varphi \text{ in } \Omega,$$

for any $i = 1, 2, ...$ Using the same reasoning as before, there exists a harmonic function $\bar{w}$ in $B_r(x_0)$ with a maximum attained at x_0. Then, by the strong maximum principle applied to $w - \bar{w}$ in $B_{r'}(x_0)$ for any $r' < r$, we deduce that $w - \bar{w}$ is constant and thus is equal to zero. This implies that $w = \bar{w}$ in $B_r(x_0)$. Furthermore, we have $w(\bar{x}) = \bar{w}(\bar{x}) = u_\varphi(\bar{x})$. Hence, $w = u_\varphi$ in $B_r(x_0)$ since $\bar{x}$ can be any element of $B_r(x_0)$. This proves that u_φ is harmonic in $B_r(x_0)$.

Lemma 4.5. *Let φ be a continuous function on $\partial\Omega$ and u_φ be the function defined in lemma 4.3. For some $x_0 \in \partial\Omega$, suppose $w_{x_0} \in C(\bar{\Omega})$ is a subhmarnoic function in Ω such that*

$$w_{x_0}(x_0) = 0, \ w_{x_0}(x) < 0 \quad \textit{for any} \quad x \in \partial\Omega \backslash \{x_0\},$$

then

$$\lim_{x \to x_o} u_\varphi(x) = \varphi(x_0).$$

Proof. As before, consider the set

$$S = \{v \in C(\bar{\Omega}) \mid v \text{ is subharmonic in } \Omega, v \leq \varphi \text{ on } \partial\Omega\}.$$

To simplify the notation, we just write $w = w_{x_0}$ and set $M = \max_{\partial\Omega}|\varphi|$. Let $\varepsilon > 0$ be arbitrary, and by continuity of φ at x_0, there exists a $\delta > 0$ such that

$$|\varphi(x) - \varphi(x_0)| < \varepsilon \text{ for any } x \in \partial\Omega \cap B_\delta(x_0).$$

We then choose K suitably large such that $-Kw(x) \geq 2M$ for any $x \in \partial\Omega \backslash B_\delta(x_0)$. Thus,

$$|\varphi(x) - \varphi(x_0)| < \varepsilon - Kw \text{ for } x \in \partial\Omega.$$

Since $\varphi(x_0) - \varepsilon + Kw(x)$ is a subharmonic function in Ω with $\varphi(x_0) - \varepsilon + Kw \leq \varphi$ on $\partial\Omega$, we have that $\varphi(x_0) - \varepsilon + Kw \in S$. The definition of u_φ then implies that

$$\varphi(x_0) - \varepsilon + KW \leq u_\varphi \quad \text{in } \Omega.$$

However, $\varphi(x_0) + \varepsilon - Kw$ is super-harmonic in Ω with $\varphi(x_0) + \varepsilon - Kw \geq \varphi$ on $\partial\Omega$. Thus, for any $v \in S$, we obtain from lemma 4.3

$$v(x) \leq \varphi(x_0) + \varepsilon - Kw(x) \text{ for } x \in \Omega. \tag{4.5}$$

Again, by the definition of u_φ,

$$u_\varphi(x) \leq \varphi(x_0) + \varepsilon - Kw(x) \text{ for } x \in \Omega. \tag{4.6}$$

Hence, equation (4.5) and equation (4.6) imply

$$|u_\varphi(x) - \varphi(x_0)| < \varepsilon - Kw(x) \text{ for } x \in \Omega,$$

and since w is continuous with $w(x) \to w(x_0) = 0$ as $x \to x_0$, we arrive at

$$\lim_{x \to x_0} \sup |u_\varphi(x) - \varphi(x_0)| < \varepsilon.$$

To finish the proof we let $\varepsilon \to 0$. $\square$

Remark 4.6. The existence of a function $w_{x_0} \in C(\overline{\Omega})$ as required by lemma 4.5 is not guaranteed for any domain Ω. However, it is guaranteed for any domain Ω that satisfies the exterior sphere condition at $x_0 \in \partial\Omega$, meaning there exists y_0 and $r > 0$, such that $\Omega \cap B_r(y_0) = \emptyset$ and $\overline{\Omega} \cap \overline{B_r(y_0)} = \{x_0\}$. In our case, we can construct w_{x_0} using Γ.

$$w_{x_0}(x) = \Gamma(x - y_0) - \Gamma(x_0 - y_0), \tag{4.7}$$

for any $x \in \overline{\Omega}$, where Γ is the fundamental solution of the Laplace equation.

To summarize, we have proven the following theorem.

Theorem 4.7 (Perron method existence theorem)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the exterior sphere condition at every boundary point. Then, for any $\varphi \in C(\partial\Omega)$, the Dirichlet problem* (4.1) *admits a solution $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$.*

4.2 Second order elliptic operators

Second order elliptic operators can come in divergence form or non-divergence form. The former lend themselves naturally to the energy method and we can establish the notion of weak solutions for them, while the latter are better approached using maximum principles.

Let Ω be an bounded open subset of $\mathbb{R}^n$, $A = A(x) = \big(a_{ij}(x)\big)_{i,j=1\dots n}$ a $n \times n$ matrix of functions, $b = b(x) = \big(b_i(x)\big)_{i=1\dots n}$ be a n-tuple of functions and $c = c(x)$ any function.

Definition 4.8. A differential operator L of second order is said to be in divergence form if

$$Lu := \operatorname{div}(A(x)\nabla u) + b(x) \cdot \nabla u + c(x)u, \tag{4.8}$$

or in non-divergence form if it can be written as

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i x_j} + b(x) \cdot \nabla u + c(x)u. \tag{4.9}$$

Remark 4.9. So far we haven't put any restrictions on a_{ij}, b_i and c. It is quite trivial to realize that the above definition in the case of the divergence form only makes sense if we impose C^1 regularity on all functions $\big(a_{ij}(x)\big)$ in the matrix $A(x)$. Doing so also allows us to switch between the two forms. By the definition of div, we have

$$\begin{aligned} \operatorname{div}(A(x)\nabla u) &= \sum_{k=1}^{n} \frac{\partial (A(x)\nabla u)_k}{\partial x_k} \\ &= \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i x_j} + \Big(\sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} \Big)_i \cdot \nabla u. \end{aligned} \tag{4.10}$$

Replacing $\operatorname{div}(A(x) \cdot \nabla u)$ with equation (4.10) in equation (4.8) gives us the non-divergence form

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i x_j} + \Big(\sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} + b_i(x) \Big)_i \cdot \nabla u + c(x)$$

Conversely, by $C^1(\Omega)$ regularity of $a_{ij}(x)$ we can transform equation (4.9) from non-divergence form to divergence form. Let

$$\tilde{b_i}(x) = b_i(x) - \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j},$$

which gives us

$$\begin{aligned}
Lu &= \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i x_j} + b(x)\nabla u + c(x)u \\
&= \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i x_j} + \Big(\sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j}\Big)_i \cdot \nabla u + \tilde{b}(x)\cdot\nabla u + c(x)u \\
&= \operatorname{div}(A(x)\nabla u) + \tilde{b}(x)\nabla u + c(x)u.
\end{aligned}$$

Definition 4.10. A linear second degree differential operator is uniformly elliptic if there exist $\alpha, \beta > 0$ such that a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$, the following holds.

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2.$$

4.3 Sobolev spaces

So far, we focused on finding solutions of C^2 regularity. In this section we will introduce the notion of weak derivatives, weak solutions and sobolev spaces, which will lay the foundation for the existence result for weak solutions in a subsequent section. For more details on Sobolev spaces, the reader may consult [6], chapter 1 and [2], which were heavily used to develop the material presented in this section.

Definition 4.11 (Weak derivative)**.** Given $u, v \in L^1_{\text{loc}}(\Omega)$ and a multi-index α. We say that v is the α-th weak partial derivative of u if

$$\int_\Omega u D^\alpha \phi dx = (-1)^\alpha \int_\Omega v\phi dx,$$

for all test functions $\phi \in C_c^\infty(\Omega)$. We denote the weak derivative by $D^\alpha u = v$.

Lemma 4.12 (Uniqueness of weak derivative)**.** *A weak α-th partial derivative of u is uniquely defined up to a set of measure zero, provided that it exists.*

Proof. Suppose that $v, \tilde{v}$ are weak derivatives of u for the same multi-index α. Then we have $\int_\Omega (v - \tilde{v})\phi dx = 0$ for all $\phi \in C_c^\infty(\Omega)$ and therefore $v = \tilde{v}$ almost everywhere. For a detailed development of the final step in this proof, i.e. concluding that $(v - \tilde{v}) = 0$ a.e. based on the fact that the above integral is 0 for all test functions ϕ you may consult [8]. You can find a full development of this result there. □

Definition 4.13. We define the sobolev space $W^{k,p}(\Omega)$ for k non-negative integer and $1 \leq p \leq \infty$ as the set of locally summable functions $u : \Omega \to \mathbb{R}$ such that for every multi-index $|\alpha| \leq k$, the weak partial derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$. Additionally we define the norm as

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \Big(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p dx\Big)^{1/p}, & (1 \leq p < \infty) \\ \sum_{|\alpha|\leq k} \text{ess sup } |D^\alpha u|. & (p = \infty) \end{cases}$$

Remark 4.14. We consider two functions u and v belonging to $W^{k,p}(\Omega)$ to be identical if they are identical almost everywhere.

Theorem 4.15 (Elementary properties of weak derivative)**.** *Suppose $u, v \in W^{k,p}(\Omega)$ and $|\alpha| \leq k$. Then*

i) $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all multindexes α, β such that $|\alpha| + |\beta| \leq k$.

ii) For each $\lambda, \mu \in \mathbb{R}$ we have $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\beta v$.

iii) If O is an open subset of Ω, then $u \in W^{k,p}(O)$.

iv) if $\varphi \in C_c^\infty(\Omega)$, then $\varphi u \in W^{k,p}(\Omega)$ and the Leibniz formula holds, i.e.

$$D^\alpha(\varphi u) = \sum_{\beta \leq \alpha} \binom{|\alpha|}{|\beta|} D^\beta \varphi D^{\alpha-\beta} u,$$

where

$$\binom{|\alpha|}{|\beta|} = \frac{|\alpha|!}{|\beta|!(|\alpha| - |\beta|)!}$$

and β is a multi-index and $\beta \leq \alpha$ is defined as

$$\beta = (\beta_1, ..., \beta_n) \leq (\alpha_1, .., \alpha_n) = \alpha \Leftrightarrow \beta_i \leq \alpha_i,$$

for all $i = 1, ..., n$.

Remark 4.16. For a full proof of theorem 4.15, the reader may consult [2], page 263.

Definition 4.17 (Convergence)**.** Given a sequence of functions $\{u_n\}_{n=1}^\infty \subset W^{k,p}(\Omega)$ and a function $u \in W^{k,p}(\Omega)$, we say that

i) $u_n \to u$ in $W^{k,p}(\Omega)$ if $\lim_{n\to\infty} \|u_n - u\|_{W^{k,p}(\Omega)} = 0$,

ii) $u_n \to u$ in $W^{k,p}_{\text{loc}}(\Omega)$ if $u_n \to u$ in $W^{k,p}(V)$ for any $V \subset\subset \Omega$.

Definition 4.18. We define $W^{k,p}_O(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

Theorem 4.19 (Sobolev spaces as function spaces)**.** *For every $k = 1, 2, ...$ and $1 \leq p \leq \infty$ the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Proof. **Step 1.** We show that $\|u\|_{W^{k,p}(\Omega)}$ is a norm. It is easy to see that $\|\lambda u\|_{W^{k,p}(\Omega)} = |\lambda|\|u\|_{W^{k,p}(\Omega)}$ holds and by the previous remark concerning equivalency of functions in sobolev spaces we have that $\|u\|_{W^{k,p}(\Omega)} = 0$ if and only if $u = 0$ almost everywhere. Now suppose $u, v \in W^{k,p}(\Omega)$. Minkowski's inequality A.3 gives us

$$\begin{aligned}
\|u + v\|_{W^{k,p}(\Omega)} &= \Big(\sum_{|\alpha|\leq k} \|D^\alpha u + D^\alpha v\|^p_{L^p(\Omega)} \Big)^{1/p} \\
&\leq \Big(\sum_{|\alpha|\leq k} \big(\|D^\alpha u\|_{L^p(\Omega)} + \|D^\alpha v\|_{L^p(\Omega)}\big)^p \Big)^{1/p} \\
&\leq \Big(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \Big)^{1/p} + \Big(\sum_{|\alpha|\leq k} \|D^\alpha v\|^p_{L^p(\Omega)} \Big)^{1/p} \\
&= \|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}(\Omega)},
\end{aligned}$$

and therefore we have successfully shown that $\|u\|_{W^{k,p}(\Omega)}$ is indeed a norm for the sobolev space $W^{k,p}(\Omega)$.

Step 2. The only remaining thing in need of a rigorous proof is completeness. Suppose $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $W^{k,p}(\Omega)$. It is trivial to see that this means that $\{D^{\alpha}u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(\Omega)$ for any $|\alpha| \le k$. By completeness of $L^p(\Omega)$ there exists a function $u_\alpha \in L^p(\Omega)$ such that $D^\alpha u_n \to u_\alpha$ in $L^p(\Omega)$. We take some $\phi \in C_c^\infty(\Omega)$. Then we have

$$\begin{aligned}\int_\Omega uD^\alpha\phi dx &= \lim_{n\to\infty}\int_\Omega u_nD^\alpha\phi dx = \lim_{n\to\infty}(-1)^{|\alpha|}\int_\Omega D^\alpha u_n\phi dx\\ &= (-1)^{|\alpha|}\int_\Omega u_\alpha\phi dx,\end{aligned}$$

which means that for any $u \in W^{k,p}(\Omega)$ we have $D^\alpha u = u_\alpha$, with $|\alpha| \le k$ and thus $u_n \to u$ in $W^{k,p}(\Omega)$. □

Remark 4.20. For any $k \in \{0, 1, 2, ...\}$ we denote the Sobolev space $W_0^{k,2}(\Omega)$ by $H_0^k(\Omega)$. This follows from the fact that $W_0^{k,2}$ is a Hilbert space for any $k \in \{0, 1, 2, ..., \}$. For more details, the reader may consult [1], chapter 8.

4.4 Weak solutions

Definition 4.21. We define the bilinear form $B[\cdot,\cdot]$ associated with the divergence form elliptic operator L (4.8) as follows:

$$B[u,v] := \int_\Omega \big(\sum_{i,j=1}^n a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b_iu_{x_i}v + cuv\big)dx.$$

Definition 4.22 (Weak solution of an Elliptic Operator)**.** A function $u \in H_0^1(\Omega)$ is called a weak solution of equation (4.8) if

$$B[u,v] = (f,v) \text{ for all } v \in H_0^1(\Omega),$$

where $B[\cdot,\cdot]$ is the bilinear form

$$B[u,v] := \int_\Omega \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_j}v + c(x)uv.$$

4.5 Existence of weak solutions

In this section, we will use the Lax-Milgram theorem to show that there exists a unique weak solution to the following Dirichlet boundary value problem. This chapter is based on the lectures held by Professor Radulescu [7]. In addition to the aforementioned lecture notes, the author made use of [2] to address gaps in his understanding of the material presented in this section.

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases} \tag{4.11}$$

Remark 4.23. As we will see later, the choice of μ is not arbitrary. As part of proving the existence of a unique weak solution to the above problem, we will also clarify the choice of μ.

4.5.1 Lax-Milgram theorem

Theorem 4.24 (Lax-Milgram)**.** *Let H be a Hilbert space with an inner-product induced norm $\|\cdot\|$ and let $B : H \times H \to \mathbb{R}$ be a bilinear form. If there exist $\alpha, \beta > 0$ such that for every $u, v \in H$ we have*

i) Boundedness: $|B[u, v]| \leq \alpha \|u\| \|v\|$,

ii) Coercivity: $\beta \|u\|^2 \leq B[u, u]$,

then for any $f \in H'$ there exists a unique $u \in H$ such that

$$B[u, v] = (f, v) \text{ for all } v \in H,$$

where $(\cdot, \cdot)$ is the inner-product associated with H and H' is the dual of H.

Proof. **Existence.** Taking a fixed $w \in H$ means that $v \mapsto B[w, v]$ is a bounded linear functional on H. By the Riesz representation theorem A.13, there exists a unique $u \in H$ such that $(u, v) = B[w, v]$ for all $v \in H$, where $(\cdot, \cdot)$ is the inner product of H. Let us now define an operator $A : H \to H$, $A[w] := u$. We show that A is a bounded linear operator. To show linearity, we note that

$$\begin{aligned}(A[\lambda_1 u_1 + \lambda_2 u_2], v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] = \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= (\lambda_1 A[u_1] + \lambda_2 A[u_2], v) \text{ for all } v \in H,\end{aligned}$$

which means that $A[\lambda_1 u_1 + \lambda_2 u_2] = \lambda_1 A[u_1] + \lambda_2 A[u_2]$. To show boundedness, we note that

$$\|Au\|^2 = (Au, Au) = B[Au, u] \leq \alpha \|u\| \cdot \|Au\|,$$

and thus $\|Au\| \leq \alpha \|u\|$. Furthermore range(A) is closed in H. Let $\{y_k\}$ be a convergent sequence in range(A) so that there exists a sequence $\{u_k\} \subset H$ for which $y_k = A[u_k] \to y \in H$. By coercivity we have $\|u_k - u_j\| \leq \beta \|A[u_k] - A[u_j]\|$, which implies that $\{u_k\}$ is a Cauchy sequence in H. Therefore u_k converges to some element $u \in H$ and $y = A[u]$ and thus we conclude that $y \in$ range(A). Therefore range(A) is closed in H. Suppose that range$(A) \neq H$. That means that we can express H as $H = \text{range}(A) \oplus \text{range}(A)^\perp$. Let $z \in$ range(A) be a non-zero element. By the coercivity condition we have $\beta \|z\|^2 \leq B[z, z] = (Az, z) = 0$, which is a contradiction. The Riesz representation theorem A.13 implies that for each $\varphi \in H'$ there exists an element $z \in H$ for which $\varphi(v) = (z, v)$ for all $v \in H$. This means that we can find a u such that $z = A[u]$, meaning $(z, v) = (Au, v) = B[u, v]$ for all $v \in H$. This means that we've found an element $u \in H$ for which $B[u, v] = (f, v)$ for all $v \in H$.

Uniqueness. Suppose u_1, u_2 both satisfy $B[u_1, v] = B[u_2, v] = (f, v)$ for all $v \in H$. Therefore, $B[u_1 - u_2, v] = 0$ for all $v \in H$. By coercivity of the bilinear form B we have $\beta \|u_1 - u_2\|^2 \leq B[u_1 - u_2, u_1 - u_2] = 0$ and thus $u_1 = u_2$. □

Theorem 4.25 (Energy estimates). *There exist constants $\alpha, \beta, \gamma > 0$, such that*

i) $|B[u,v]| \leq \alpha \|u\|_H \|v\|_H$,

ii) $\beta \|u\|_H^2 \leq B[u,u] + \gamma \|u\|_{L^2(\Omega)}^2$ *for all* $u, v \in H$.

Proof. To prove the first claim of the theorem we start out by establishing a boundary for $|B[u,v]|$.

$$\begin{aligned} |B[u,v]| &= \Big| \int_\Omega \sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x) u_{x_i} v + cuv dx \Big| \\ &\leq \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} \int_\Omega |Du \cdot Dv| dx \\ &+ \sum_{i=1}^n \|b_i\|_{L^\infty} \int_\Omega |Du||v| dx \\ &+ \|c\|_{L^\infty} \int_\Omega u^2 dx, \end{aligned} \tag{4.12}$$

where we justify the inequality by the fact that a_{ij}, b_i, c are by hypothesis in $L^\infty(\Omega)$. Using the Hölder inequality A.2 as many times as required and by the definition of the norm associated with H we arrive at

$$|B[u,v]| \leq C \|u\|_H \|v\|_H,$$

where C is a constant.

To prove the second claim of the theorem, we use the definition of ellipticity. There exists $\lambda > 0$, such that

$$\begin{aligned} \lambda \int_\Omega |Du|^2 dx &\leq \int_\Omega \sum_{i,j=1}^n a_{ij}(x) u_{x_i} u_{x_j} dx \\ &= B[u,u] - \int_\Omega \sum_{i=1}^n b_i(x) u_{x_i} u + cu^2 dx \\ &\leq B[u,u] + \sum_{i=1}^n \|b_i\|_{L^\infty} \int_\Omega |Du||u| dx + \|c\|_{L^\infty} \int_\Omega u^2 dx. \end{aligned} \tag{4.13}$$

For any $\varepsilon > 0$ and x, y we have

$$xy \leq \varepsilon x^2 + \frac{y^2}{4\varepsilon},$$

since

$$0 \leq (\sqrt{\varepsilon} x - \frac{y}{2\sqrt{\varepsilon}})^2 \quad \Leftrightarrow \quad xy \leq \varepsilon x^2 + \frac{y^2}{4\varepsilon}.$$

Using this simple fact, we have

$$|Du||u| \leq \varepsilon|Du|^2 + \frac{u^2}{4\varepsilon} \Rightarrow \int_\Omega |Du||u|dx \leq \varepsilon \int_\Omega |Du|^2 dx + \frac{1}{4\varepsilon} \int_\Omega u^2 dx.$$

Choosing $\varepsilon > 0$ such that $\varepsilon \sum_{i=1}^{n} \|b_i\|_{L^\infty} < \frac{\lambda}{2}$ and putting this into equation (4.13) yields

$$\begin{aligned} \lambda \int_\Omega |Du|^2 dx &\leq B[u,u] + \big(\sum_{i=1}^{n} \|b_i\|_{L^\infty}\big)\big(\varepsilon \int_\Omega |Du|^2 dx + \frac{1}{4\varepsilon} \int_\Omega u^2 dx\big) \\ &\quad + \|c\|_{L^\infty} \int_\Omega u^2 dx \\ &\leq B[u,u] + \frac{\lambda}{2} \int_\Omega |Du|^2 dx + \frac{1}{4\varepsilon}\big(\sum_{i=1}^{n} \|b_i\|_{L^\infty}\big) + \|c\|^{L^\infty} \int_\Omega u^2 dx. \end{aligned}$$

A simple rearrangement of terms and adding $\frac{\lambda}{2} \int_\Omega |u|^2 dx$ to both sides of the inequality, results in

$$\frac{\lambda}{2}\|u\|_H^2 \leq B[u,u] + (C + \tfrac{\lambda}{2})\|u\|^2_{L^2(\Omega)}.$$

□

Theorem 4.26 (First existence theorem for weak solutions using the Lax-Milgram theorem)**.** *There exists a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function $f \in L^2(\Omega)$, there exists a unique weak solution $u \in H = H_0^2(\Omega)$ of the following Dirichlet boundary value problem:*

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega, \\ \qquad\quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Let γ be the constant from theorem 4.25. We choose $\mu \geq \gamma$ and define the bilinear form $B_\mu[u,v] := B[u,v] + \mu(u,v)_{L^2}$, for any $u, v \in H$. We will show that this bilinear form satisfies the hypothesis of the Lax-Milgram theorem 4.24. By theorem 4.25 and the Cauchy-Schwarz inequality A.1 we have

$$\begin{aligned} |B_\mu[u,v]| &= |B[u,v] + \mu(u,v)_{L^2}| \leq |B[u,v]| + \mu|(u,v)_{L^2}| \\ &\leq C\|u\|_H\|v\|_H + \mu\|u\|_{L^2}\|v\|_{L^2}. \end{aligned} \tag{4.14}$$

Since $H = H_0^2(\Omega) = W_0^{2,2}(\Omega)$, we have

$$\|u\|_{H(\Omega)} = \Big(\sum_{|\alpha|\leq 2} \int_\Omega |D^\alpha u|^2 dx\Big)^{1/2} = \Big(\|u\|^2_{L^2} + \sum_{1\leq|\alpha|\leq 2} \int_\Omega |D^\alpha u|^2 dx\Big)^{\frac{1}{2}}.$$

We can rewrite equation (4.14) by using the fact that the square root is concave:

$$\begin{aligned}
|B_\mu[u,v]| &\le C\|u\|_H\|v\|_H + \mu\|u\|_{L^2}\|v\|_{L^2} \\
&= \Big[C^2\big(\|u\|_{L^2}^2 + \sum_{|\alpha|=1}^{2} \|D^\alpha u\|_{L^2}^2\big)\big(\|v\|_{L^2}^2 + \sum_{|\alpha|=1}^{2} \|D^\alpha v\|_{L^2}^2\big)\Big]^{\frac{1}{2}} \\
&\quad + \Big[\mu^2(\|u\|_{L^2}^2\|v\|_{L^2}^2)\Big]^{\frac{1}{2}} \\
&\le \big[\big(C^2 + 2\mu^2\big)\big(\|u\|_{L^2}^2 + \sum_{|\alpha|=1}^{2} \|D^\alpha u\|_{L^2}^2\big)\big(\|v\|_{L^2}^2 + \sum_{|\alpha|=1}^{2} \|D^\alpha v\|_{L^2}^2\big)\big]^{\frac{1}{2}} \\
&\le (C^2 + \mu^2)^{\frac{1}{2}}\|u\|_H\|v\|_H.
\end{aligned}$$

Using the second bound from theorem 4.25, we find that

$$B_\mu[u,u] = B[u,u] + \mu(u,u)_{L^2} \ge B[u,u] + \|u\|_{L^2}^2 \ge \beta\|u\|_H,$$

which establishes coercivity. Let f be a function in $L^2(\Omega)$ and define $\varphi_f(v) = (f,v)_{L^2}$. Using the Cauchy-Schwarz inequality A.1 we show that φ_f is a bounded linear functional.

$$|\varphi_f(v)| = |(f,v)_{L^2}| \le \|f\|_{L^2}\|v\|_{L^2} \le \|f\|_{L^2}\|v\|_H.$$

We established coercivity and boundedness of the bilinear form B_μ and showed that the linear functional φ_f is bounded. Therefore, having fulfilled the hypothesis of the Lax-Milgram theorem 4.24 we conclude that there exists a unique function $u \in H$ such that $B_\mu[u,v] = \varphi_f(v)$ for all $v \in H$. In other words, there exists a **unique** weak solution $u \in H$ to the Dirichlet boundary value problem (4.11). □

A Elementary results

Theorem A.1 (Cauchy-Schwarz inequality)**.** *Let $(V,(\cdot,\cdot))$ be an inner product space. Then $|(u,v)|^2 \le (u,u)\cdot(v,v)$ holds for all $u,v \in V$, where $(\cdot,\cdot)$ denotes the inner product of V.*

Theorem A.2 (Hölder inequality)**.** *Let $p,q \in [1,\infty]$, such that $\frac{1}{p}+\frac{1}{q} = 1$ and $\Omega \subset \mathbb{R}^n$. For any two functions $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, we have $fg \in L^1(\Omega)$ and additionally the inequality $\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}$ holds.*

Theorem A.3 (Minkowski inequality)**.** *Let f and g be two functions such that $f,g \in L^p(\Omega)$, where $\Omega \subset \mathbb{R}^n$ and $1 \le p \le \infty$. Then we have $f+g \in L^p(\Omega)$ and the inequality $\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$ holds.*

Remark A.4. For more information on the Hölder inequality and the Minkowski inequality, the reader may consult [9], chapter 1.

Theorem A.5 (Fubini's theorem)**.** *Let X and Y be two σ-finite measure spaces and $f : X \times Y \to \mathbb{R}$ a function that is $X \times Y$ integrable. Then*

$$\int_X \Big(\int_Y f(x,y)dy\Big)dx = \int_Y \Big(\int_X f(x,y)dx\Big)dy = \int_{X\times Y} f(x,y)d(x,y).$$

Remark A.6. For more details, the reader may consult [10], page 75.

Theorem A.7 (Green's formulas)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^1 boundary and $u, v \in C^2(\bar{\Omega})$. Then*

i) $\int\limits_{\Omega} \Delta u dx = \int\limits_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma,$

ii) $\int\limits_{\Omega} Dv \cdot Du dx = -\int\limits_{\Omega} u \Delta v dx + \int\limits_{\partial\Omega} \frac{\partial v}{\partial \nu} u d\sigma,$

iii) $\int\limits_{\Omega} u\Delta v - v\Delta u dx = \int\limits_{\partial\Omega} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} d\sigma.$

Remark A.8. For more details, the reader may consult [2], page 715.

Theorem A.9 (Gauss divergence theorem)**.** *Given a compact subset V of $\mathbb{R}^n$ having a piecewise smooth boundary S and a continuously differentiable vector field F, that is defined in a neighbourhood of V, we have:*

$$\iiint\limits_V (\nabla \cdot \mathbf{F}) dV = \oiint\limits_S (\mathbf{F} \cdot \mathbf{n}) dS.$$

Remark A.10. For more details, the reader may consult [4], page 60, for a very general version of the Stokes theorem.

Theorem A.11 (Lebesgue differentiation theorem)**.** *For a Lebesgue integrable function, i.e. $f \in L^1(\mathbb{R}^n)$, we have*

$$\lim_{r \to 0^+} \frac{1}{\lambda(B(x,r))} \int\limits_{B(x,r)} |f(t) - f(x)| d\lambda(t) = 0 \quad \textit{for almost all} \quad x \in \mathbb{R}^n,$$

where $B(x,r)$ is a ball in $\mathbb{R}^n$ of radius r, centered in x and λ is the Lebesgue measure.

Remark A.12. For more details, the reader may consult [10], page 104.

Theorem A.13 (Riesz representation theorem)**.** *Let H be a Hilbert space and φ a continuous linear functional, i.e. $\varphi \in H'$, where H' is the dual space of H. Then there exists $f \in H$ such that for any $x \in H$, $\varphi(x) = (f, x)$ and $\|f\|_H = \|\varphi\|_{H'}$, where $(\cdot, \cdot)$ denotes the inner product of the H.*

Remark A.14. For more details, the reader may consult [1], page 97.

Lemma A.15 (Boundary of connected components)**.** *For any connected component V of a set $\Omega \subset \mathbb{R}^n$, we have $\partial V \subset \partial\Omega$.*

Proof. Let x be any element of $V \cap \text{int}(\Omega)$. Since $\mathbb{R}^n$ is locally connected, there exists a connected neighbourhood of x in $\text{int}(\Omega)$, i.e. there exists $U_x \subset \text{int}(\Omega)$ such that $x \in U_x$. Since U_x and V are connected sets and are non-disjoint, their union must be connected. Suppose that U_x is not a subset of V. This implies that V is not a maximally connected subset of Ω, which is in direct contradiction with the assumption that V is a connected component of Ω. Therefore, $x \in U_x \subset V$ and V is relatively open in Ω. V is a connected component and therefore relatively closed in Ω. Thus, $\text{int}(V) = V = \overline{V} \cap \text{int}(\Omega)$. Since $x \in \overline{V} \cap \text{int}(\Omega) \Rightarrow x \in \text{int}(V)$, we can show that $\partial V \subset \partial\Omega$ as follows:

$$\partial V = \overline{V} \setminus \text{int}(V) \subset \overline{V} \setminus \text{int}(\Omega) = \overline{\Omega} \setminus \text{int}(\Omega) = \partial\Omega.$$

More information on connected components, their properties and the notion of connectedness can be found in [5], page 157. □

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