MATH 8450

de Rham's theorem

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Our aim is to work towards a proof of the de Rham theorem. We will assume that the reader has some basic knowledge of sheaves and sheaf cohomology.

The development in this document follows the lecture notes of a course titled "Basic Sheaf Theory" given by Professor Kucharz at Jagiellonian University in the Fall 2020. During this course, Professor Kucharz made a passing remark that with what has been shown during the course, it should be trivial to show de Rham's theorem. The aim of this work is to put together the relevant pieces to get a proof of de Rham's theorem.

Definition 1. A sheaf $\mathcal{F}$ on X is said to be acyclic if $H^q(X, \mathcal{F}) = 0$ for all $q \geq 1$.

Definition 2. A sheaf $\mathcal{F}$ on X is said to be flabby if the restriction morphism $F(X) \to F(U)$ is surjective for any $U \subset X$ open.

Definition 3. A resolution of a sheaf $\mathcal{F}$ on a topological space X is an exact sequence $0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$ of sheaves on X.

Definition 4. We define the sheaf $\underline{\mathbb{R}}$ as the sheafification of the constant presheaf $\mathbb{R}$, which assigns to each open set U, the set $\mathbb{R}$ and whose restriction maps are given by the identity function.

Theorem 5

Let

$$0 \to \mathcal{F} \xrightarrow{i} \mathcal{F}^0 \xrightarrow{\varphi^0} \mathcal{F}^1 \xrightarrow{\varphi^1} \mathcal{F}^2 \xrightarrow{\varphi^2} \dots$$

be an acyclic resolution of a sheaf $\mathcal{F}$ on X, that is to say that $\mathcal{F}^p$ is acyclic for any $p \geq 0$. Then the cohomology group $H^k(X, \mathcal{F})$ is isomorphic to the k-th cohomology group of the cochain complex

$$\dots \to 0 \to \Gamma(X, \mathcal{F}^0) \xrightarrow{\Gamma(\varphi^0)} \Gamma(X, \mathcal{F}^1) \xrightarrow{\Gamma(\varphi^1)} \Gamma(X, \mathcal{F}^2) \xrightarrow{\Gamma(\varphi^2)} \dots$$

of Abelian groups. That is,

$$\begin{aligned} H^0(X, \mathcal{F}) &\simeq \ker \Gamma(\varphi^0), \\ H^k(X, \mathcal{F}) &\simeq \ker \Gamma(\varphi^k)/\operatorname{im} \Gamma(\varphi^{k-1}) \quad \text{for } k \geq 1. \end{aligned}$$

Proof. The sequence of sheaves $0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1$ is exact, thus so is the sequence of groups $0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1)$. This implies $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) \simeq \ker \Gamma(\varphi^0)$.

Suppose now that $k \geq 1$. For the subsheaf $\mathcal{G} := \operatorname{im} \varphi^0 = \ker \varphi^1$ of $\mathcal{F}^1$, we have the following two exact sequences.

$$\begin{aligned} &0 \to \mathcal{F} \to \mathcal{F}^0 \xrightarrow{\psi} \mathcal{G} \to 0, \\ &0 \to \mathcal{G} \hookrightarrow \mathcal{F}^1 \to \mathcal{F}^2 \to \dots, \end{aligned}$$

where ψ is induced by φ^0 and $\mathcal{G} \hookrightarrow \mathcal{F}^1$ is the inclusion. The first exact sequences implies that

$$\Gamma(X, \mathcal{F}^0) \xrightarrow{\Gamma(\psi)} \Gamma(X, \mathcal{G}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}^0) = 0$$

is exact and therefore $H^1(X, \mathcal{F}) \simeq \Gamma(X, \mathcal{G})/\operatorname{im} \Gamma(\psi)$. By the second exact sequence, $0 \to \Gamma(X, \mathcal{G}) \hookrightarrow \Gamma(X, \mathcal{F}^1) \xrightarrow{\Gamma(\varphi^1)} \Gamma(X, \mathcal{F}^2)$ is exact, so $\Gamma(X, \mathcal{G}) = \ker \Gamma(\varphi^1)$. Since $\mathcal{G} = \operatorname{im} \varphi^0$, we have $\operatorname{im} \Gamma(\psi) = \operatorname{im} \Gamma(\varphi^0)$. Thus $H^1(X, \mathcal{F}) \simeq \ker \Gamma(\varphi^1)/\operatorname{im} \Gamma(\varphi^0)$.

For $k \geq 2$, the first exact sequence yields another exact sequence $0 = H^{k-1}(X, \mathcal{F}^0) \to H^{k-1}(X, \mathcal{G}) \to H^k(X, \mathcal{F}) \to H^k(X, \mathcal{F}^0) = 0$, thus $H^{k-1}(X, \mathcal{G}) \simeq H^k(X, \mathcal{F})$. We proceed by induction on k. The second exact sequence yields $H^{k-1}(X, \mathcal{G}) \simeq \ker \Gamma(\varphi^k)/\operatorname{im} \Gamma(\varphi^{k-1})$ and we are done by virtue of the two isomorphisms that we have established. ∎

Proposition 6

Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G}$ be an exact sequence of sheaves on X. Then, for an arbitrary subset A of X, the induced sequence $0 \to \Gamma(A, \mathcal{E}) \overset{\Gamma(A,\varphi)}{\to} \Gamma(A, \mathcal{F}) \overset{\Gamma(A,\psi)}{\to} \Gamma(A, \mathcal{G})$ is exact.

Proposition 7

Let $\mathcal{F}$ be a sheaf on X and $s \in \Gamma(A, \mathcal{F})$. Suppose that X is paracompact. Then there exists an open neighborhood $U \subset X$ of A and a section $t \in \Gamma(U, \mathcal{F})$ such that $t|_A = s$.

Definition 8. A sheaf $\mathcal{F}$ on a topological space X is called soft if for each closed subset A of X the restriction morphism $\Gamma(X, \mathcal{F}) \to \Gamma(A, \mathcal{F})$, $s \mapsto s|_A$ is surjective.

Theorem 9

Assume that X is a paracompact topological space. Then each flabby sheaf $\mathcal{F}$ on X is soft.

Proof. Let A be a closed subset of X and $s \in \Gamma(A, \mathcal{F})$. By Proposition 7 there exists an open neighborhood $U \subset X$ such that s is in the image of $\Gamma(U, \mathcal{F}) \to \Gamma(A, \mathcal{F})$. Since $\mathcal{F}$ is flabby, $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is surjective. ∎

Theorem 10

Assume that X is paracompact. Let $0 \to \mathcal{E} \overset{\varphi}{\to} \mathcal{F} \overset{\psi}{\to} \mathcal{G} \to 0$ be a short exact sequence with $\mathcal{E}$ soft. Then $0 \to \Gamma(X, \mathcal{E}) \overset{\Gamma(\varpi)}{\to} \Gamma(X, \mathcal{F}) \overset{\Gamma(\psi)}{\to} \Gamma(X, \mathcal{G}) \to 0$ is exact.

Proof. It suffices to show that $\Gamma(\psi)$ is surjective. Let $t \in \Gamma(X, \mathcal{G})$. Since $\psi : \mathcal{F} \to \mathcal{G}$ is surjective, there is an open cover $\{U_i\}_{i \in I}$ of X and sections $\{s_i \in \Gamma(U_i, \mathcal{F})\}_{i \in I}$ such that $\psi(s_i) = t|_{U_i}$. Since X is paracompact, the open cover can be assumed to be locally finite. Thus there is an open cover $\{V_i\}_{i \in I}$ of X such that $\bar{V_i} \subset U_i$ for all $i \in I$. Consider the set $\mathcal{X}$ of all pairs (A, s) where A is a subset of X that is the union of $\bar{V_i}$-s and $s \in \Gamma(A, \mathcal{F})$ such that $\psi(s) = t|_A$. A is closed, $\mathcal{X}$ is non-empty and partially ordered under the relation $(A_1, s_1) \leq (A_2, s_2)$ if $A_1 \subset A_2$ and $s_1 = s_2|_{A_1}$ and $\mathcal{X}$ has an upper bound in $\mathcal{X}$. Using Zorn's lemma we find a maximal element (A, s). If $A = X$, we are done. Otherwise, there is $i_0 \in I$ such that $\bar{V_{i_0}}$ is not contained in A. By construction, $\psi(s|_{A \cap \bar{V_{i_0}}} - s_{i_0}|_{A \cap \bar{V_{i_0}}}) = 0$. By Proposition 6, the induced sequence of Abelian groups

$$0 \to \Gamma(A \cap \bar{V_{i_0}}, \mathcal{E}) \to \Gamma(A \cap \bar{V_{i_0}}, \mathcal{F}) \to \Gamma(A \cap \bar{V_{i_0}}, \mathcal{G})$$

is exact, there exists a section $u_0 \in \Gamma(A \cap \bar{V_{i_0}}, \mathcal{E})$ with $u|_{A \cap \bar{V_{i_0}}} = u_0$. Then the section s and $s_{i_0}|_{\bar{V_{i_0}}} + \varphi(u)|_{\bar{V_{i_0}}}$ agree on $A \cap \bar{V_{i_0}}$ and hence they define a section $s' \in \Gamma(A \cap \bar{V_{i_0}}, \mathcal{F})$ with $s'|_A = s$. But this contradicts the maximality of (A, s), thus we have a contradiction. ∎

Theorem 11

Suppose that X is paracompact. Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be exact. Let $\mathcal{E}$ and $\mathcal{F}$ be soft sheaves. Then $\mathcal{G}$ is soft.

Proof.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X,\mathcal{E}) & \longrightarrow & \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{G}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(A,\mathcal{E}) & \longrightarrow & \Gamma(A,\mathcal{F}) & \longrightarrow & \Gamma(A,\mathcal{G}) & \longrightarrow & 0 \end{array}$$

is commutative. By Theorem 10 the first row is exact. The second row is exact by a similar reasoning. The middle vertical morphism is surjective since $\mathcal{F}$ is soft. Thus $\Gamma(X,\mathcal{G}) \to \Gamma(A,\mathcal{G})$ is surjective. Thus $\mathcal{G}$ is soft since A is arbitrary. ∎

Theorem 12

Let X be a paracompact topological space. Then each soft sheaf $\mathcal{F}$ on X is acyclic.

Proof. Let $\overline{\mathcal{F}} = \mathcal{DF}/\mathcal{F}$ be the quotient sheaf. We have a canonical short exact sequence $0 \to \mathcal{F} \to \mathcal{DF} \to \overline{\mathcal{F}} \to 0$. We know that $\mathcal{DF}$ is flabby, thus soft by Theorem 9. Hence $\overline{\mathcal{F}}$ is soft by Theorem 11. By Theorem 10, the induced sequence $0 \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{DF} \to \Gamma(X,\overline{\mathcal{F}}) \to 0$ is exact. On the other hand, $0 \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{DF}) \to \Gamma(X,\overline{\mathcal{F}}) \to H^1(X,\mathcal{F}) \to 0$ is exact. Therefore $H^1(X,\mathcal{F}) = 0$. If $q > 0$, then $0 \to H^q(X,\overline{\mathcal{F}}) \to H^{q+1}(X,\mathcal{F}) \to 0$ is exact. By induction we find $H^k(X,\mathcal{F}) = 0$ for $k \geq 2$. ∎

Definition 13. A sheaf $\mathcal{F}$ on X is fine if for every locally finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$ if X there exists a collection of morphisms $\{\varphi_i : \mathcal{F} \to \mathcal{F}\}_{i \in I}$ such that

1. For every $i \in I$ there exists a closed subset A_i of X such that $A_i \subset U_i$ and $(\varphi_i)_x = 0$ for all $x \in X \setminus A_i$.
2. $\sum_{i \in I} (\varphi_i)_x = \mathrm{id}_{\mathcal{F}_x}$ for all $x \in X$.

Theorem 14

Assume that X is a paracompact topological space and let $\mathcal{F}$ be a fine sheaf on X. Then $\mathcal{F}$ is a soft sheaf.

Proof. Let A be closed and let $s \in \Gamma(A,\mathcal{F})$. There exists a locally finite open cover of X and a collection of sections such that $s_i|_{A \cap U_i} = s|_{A \cap U_i}$ for all $i \in I$. Let φ be the collection of morphisms from the definition of a fine sheaf. By the definition we have A_i such that $A_i \subset U_i$ and $(\varphi_i)_x = 0$ for $x \in X \setminus A_i$. Thus $\varphi_i(s_i) = \Gamma(U_i,\varphi_i)(s_i) \in \Gamma(U_i,\mathcal{F})$ is the restriction of global sections $t_i \in \Gamma(X,\mathcal{F})$ defined by $t_i(x) = \begin{cases} (\varphi_i)_x((s_i)_x) & \text{for } x \in U_i, \\ 0 & \text{for } x \in X \setminus A_i. \end{cases}$ By construction $t = \sum_{i \in I} t_i$ is a well-defined section in $\Gamma(X,\mathcal{F})$ with $t|_A = s$. Therefore $\mathcal{F}$ is a soft sheaf. ∎

Definition 15. Let $\mathcal{A}$ be a sheaf of commutative rings with 1 on a topological space X. A sheaf $\mathcal{F}$ of Abelian groups on X is a sheaf of $\mathcal{A}$-modules if for every open set $U \subset X$ the group $\mathcal{F}(U)$ has an $\mathcal{A}(U)$-module structure and for all $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structure.

Definition 16. Let $\mathcal{A}$ be a sheaf of rings and let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules on a topological space X. We say that $\mathcal{F}$ is a fine sheaf of $\mathcal{A}$-modules if $\mathcal{F}$ is a fine sheaf when regarded as a sheaf of Abelian groups.

Proposition 17

Let $\mathcal{A}$ be a sheaf of rings (commutative with 1) and let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules on a topological space X. Assume that $\mathcal{A}$ is a fine sheaf. Then so is $\mathcal{F}$.

Proof. Let $\{U_i\}_{i\in I}$ be a locally finite open cover of X. $\mathcal{A}$ is a fine sheaf, so we have morphisms of sheaves of Abelian groups $\varphi_i : \mathcal{A} \to \mathcal{A}$ for $i \in I$ and closed subsets A_i of X such that $A_i \subset U_i$ for all $i \in I$, $(\varphi_i)_x = 0$ for any $x \in X \setminus A_i$ and $\sum_{i\in I} (\varphi_i)_x = \mathrm{id}_{\mathcal{A}_x}$ for all $x \in X$. Let $\psi_i : \mathcal{F} \to \mathcal{F}$ be as follows. If $U \subset X$ is open and $s \in \mathcal{F}(U)$, then $(\psi_i)_U(s) := (\varphi_i)_U(1)s$, where $1 \in \mathcal{A}(U)$. By construction, $(\psi_i)_x = 0$ for all $x \in X \setminus A_i$ and $\sum_{i\in I} (\psi_i)_x = \mathrm{id}_{\mathcal{F}_x}$ for all $x \in X$. Thus $\mathcal{F}$ is a soft sheaf. ∎

Theorem 18

Let X be a C^∞ manifold. Then for each non-negative integer p the sheaf $\mathcal{A}^p_X$ of differential forms of order p on X is fine.

Proof. For each open cover of X there exists a C^∞ partition of unity subordinate to the cover. By Proposition 17, $\mathcal{A}^p_X$ is a fine sheaf. ∎

Corollary 19

With the notation as in Theorem 18, $\mathcal{A}^p_X$ is an acyclic sheaf.

Proof. Follows from Theorems 18, 12 and 14. ∎

Theorem 20

Let X be a C^∞ manifold. Then there is a canonical resolution for the constant sheaf $\underline{\mathbb{R}}$ on X

$$0 \to \underline{\mathbb{R}} \xrightarrow{\epsilon} \mathcal{A}^0_X \xrightarrow{d} \mathcal{A}^1_X \xrightarrow{d} \mathcal{A}^2_X \xrightarrow{d} \dots,$$

where $\epsilon : \underline{\mathbb{R}} \to \mathcal{A}^0_X$ is the canonical embedding and $d : \mathcal{A}^p_X \to \mathcal{A}^{p+1}_X$ is the exterior differentiation.

Proof. The theorem follows from the exactness of the sequence of sheaves in the statement. Since this is a local problem, the assertion follows from the Poincare lemma. ∎

Theorem 21 (de Rham theorem)

Let X be a C^∞ manifold. For each integer q, the cohomology group $H^q(X, \underline{\mathbb{R}})$ is isomorphic to the q-th cohomology group of the cochain complex

$$\ldots \to 0 \to \mathcal{A}_X^0(X) \xrightarrow{d} \mathcal{A}_X^1(X) \xrightarrow{d} \mathcal{A}_X^2(X) \xrightarrow{d} \ldots$$

Proof. By Corollary 19 and Theorem 20, the constant sheaf $\underline{\mathbb{R}}$ on X admits an acyclic resolution

$$0 \to \underline{\mathbb{R}} \xrightarrow{\epsilon} \mathcal{A}_X^0 \xrightarrow{d} \mathcal{A}_X^1 \xrightarrow{d} \ldots .$$

Hence the conclusion follows from Theorem 5, since $\mathcal{A}_X^p(X)$ is the same as $\Gamma(X, \mathcal{A}_X^p)$, i.e. the global sections of the sheaf $\mathcal{A}_X^p$. ∎