Categories, Functors and Presheaves

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ncatlab.org: "Category theory is a toolset for describing the general abstract structures in mathematics"

Paradigm

In contrast to set theory, Category Theory does not focus on elements (called objects). Instead we are primarily interested in relations between these objects, i.e. morphisms between objects

$$x \xrightarrow{f} y.$$

Categories, Functors and Natural transformations

One particularly elegant aspect of Category theory is that it reflects on itself. Categories are primariliy about collections of morphisms. Just as morphisms are relations between objects, there are self-evident morphisms between categories, which are called functors. And there are self-evident morphisms between functors, which are called natural transformations.

Categories, Functors and Natural transformations

Category theory focuses on these three concepts

- Categories
- Functors
- Natural transformations

Higher category theory extends this idea to k-transfors for all $k \in \mathbb{N}$.

Functor category

We will encounter this notion of reflecting on itself when we define functors rigorously: Functors between two categories form a category, the so called functor category.

Moreover, the category of presheaves is a particular example of a functor category and as we will see defining presheaves in a category theoretical setting turns out to be extremely simple.

Definition: Category

A category C consists of

- a class $\mathrm{obj}(\mathrm{C})$ of objects
- a class $\hom(\mathrm{C})$ of morphisms (or arrows, or maps) between the objects. Each morphism f has a source object a and a target object b where $a, b \in \mathrm{obj}(\mathrm{C})$. We write $f : a \to b$ and we say "f is a morphism from a to b". We write $\hom(a, b)$ (or $\hom_{\mathrm{C}}(a, b)$ if there may be confusion about to which category we are referring to) to denote the hom-class of all morphisms from a to b.
- for every three objects a, b, c a binary operation $\hom(a, b) \times \hom(b, c) \to \hom(a, c)$ called composition of morphisms, such that the following axioms hold:
  - associativity: If $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$.
  - identity: For every object x, there exists a morphism $1_x : x \to x$ (sometimes denoted by id_x) called the identity morphism for x, such that for every morphism $f : a \to x$ and every morphism $g : x \to b$, we have $1_x \circ f = f$ and $g \circ 1_x = g$.

Example: Set

The category of sets is the category whose

- objects are sets,
- morphisms between sets a and b are functions from a to b,
- the composition of morphisms is given by the standard composition of functions.

We usually denote this category by Set

Example: Ab

The category of abelian groups is the category whose

- objects are abelian groups,
- morphisms are group homomorphisms.

We usually denote this category by Ab

Example: Top

The category of topological spaces is the category whose

- objects are topological spaces,
- morphisms are continuous maps between topological spaces.

We usually denote this category by Top

Example: Top_*

The category of pointed topological spaces is the category whose

- objects are pointed topological spaces, i.e. a pair (X, x) where X is a topological space and $x \in X$ is the chosen basepoint,
- morphisms are continuous basepoint preserving maps between pointed topological spaces.

We usually denote this category by Top_*.

Example: $\mathrm{Vect}_{\mathbb{F}}$

The category of vector spaces over a field $\mathbb{F}$ is the category whose

- objects are vector spaces over the field $\mathbb{F}$,
- morphisms are linear maps.

We usually denote this category by $\mathrm{Vect}_{\mathbb{F}}$

Definition: Opposite Category

The opposite or dual category C^{op} of a given category C is formed by reversing the morphisms.

To be precise, we set $\mathrm{obj}(\mathrm{C}^{\mathrm{op}}) = \mathrm{obj}(\mathrm{C})$ and $\mathrm{hom}_{\mathrm{C}^{\mathrm{op}}}(x, y) = \mathrm{hom}_{\mathrm{C}}(y, x)$ for x, y objects in C.

The composition of morphisms is given by $g \circ^{\mathrm{op}} f = f \circ g$. We usually write $\circ$ instead though.

Verifying that this is a category is not hard. We just have to show that the composition in the opposite category is associative. Everything else follows directly from C being a category.

Let $f \in \hom_{\mathrm{C}^{\mathrm{op}}}(a, b)$, $g \in \hom_{\mathrm{C}^{\mathrm{op}}}(b, c)$ and $h \in \hom_{\mathrm{C}^{\mathrm{op}}}(c, d)$. Then

$$h \circ^{\mathrm{op}} (g \circ^{\mathrm{op}} f) = (f \circ g) \circ h = f \circ (g \circ h) = (g \circ h) \circ^{\mathrm{op}} f = (h \circ^{\mathrm{op}} g) \circ^{\mathrm{op}} f.$$

Definition: epimorphism, monomorphism and isomorphism

A morphism $f\colon a \to b$ from objects a to b in a category C is called a

- monomorphism if for any pair of morphisms $g_1, g_2 : c \to a$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.
- epimorphism if $f^{\mathrm{op}} : b^{\mathrm{op}} \to a^{\mathrm{op}}$ is a monomorphism in C^{op}.
- isomorphism if there exists $g\colon b \to a$ such that $f \circ g = id_b$ and $g \circ f = id_a$.

Definition: Covariant Functor

Let C and D be categories.
A covariant functor F from C to D is a mapping that

- sends each object x in C to an object $F(x)$ in D,
- sends each morphism $f\colon x \to y$ between two objects x, y in C to a morphism $F(f) : F(x) \to F(y)$ in D such that the following conditions hold:
  - $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ for every object x in C,
  - $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f\colon x \to y$ and $g : y \to z$ in C.

In other words, covariant functors must preserve identity morphisms and composition of morphisms.

Definition: Contravariant Functors

Let C and D be categories. A contravariant functor F from C to D is a mapping that

- sends each object x in C to an object $F(x)$ in D,
- sends each morphism $f\colon x \to y$ between two objects x, y in C to a morphism $F(f) : F(y) \to F(x)$ in D such that the following conditions hold:
  - $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ for every object x in C,
  - $F(g \circ f) = F(f) \circ F(g)$ for all morphisms $f\colon x \to y$ and $g : y \to z$ in C.

A functor is by default assumed to be covariant.

We also can always transform a contravariant functor $F : \mathrm{C} \to \mathrm{D}$ into a covariant functor $G : \mathrm{C}^{\mathrm{op}} \to \mathrm{D}$.

Whether we make use of contravariant functors or rely solely on covariant functors is a question of preference.

Example: Dual Space

We have a contravariant functor $F\colon \mathit{Vect}_{\mathbb{F}} \to \mathit{Vect}_{\mathbb{F}}$ that sends each vector space V over a field $\mathbb{F}$ to its dual space V^* and each linear map to its transpose.

Example: Fundamental group

We have a functor $F\colon \mathrm{Top.} \to \mathrm{Grp}$ that sends each pointed topological space (X, x_0) to its fundamental group $\pi_1(X, x_0)$ and each morphism in $\mathrm{Top.}$ $f\colon (X, x_0) \to (Y, y_0)$ (basepoint preserving continuous maps between pointed topological spaces) to a group homomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$

Example: Lie groups and Lie algebras

The mapping that sends every real/complex Lie group to its real/complex Lie algebra is a functor.

Hom Functor

For all objects a, b in a locally small category C we have for example the following two functors from the category C to Set:

- $\mathrm{Hom}(a, \cdot) : \mathrm{C} \to \mathrm{Set}$,
- $\mathrm{Hom}(\cdot, b) : \mathrm{C} \to \mathrm{Set}$.

The first is covariant, the second contravariant.

Hom Functor

- $\mathrm{Hom}(a, \cdot)$ assigns each $b \in \mathrm{C}$ the set $\mathrm{hom}(a, b)$.
- For a morphism $f : b \to c$, we define $\mathrm{Hom}(a, f)$ as the function

$$\mathrm{Hom}(a, f) : \mathrm{hom}(a, b) \to \mathrm{hom}(a, c),$$

given by

$$g \mapsto f \circ g \text{ for each } g \in \mathrm{hom}(a, b).$$

Hom Functor

- identity

$$\mathrm{Hom}(a, \mathrm{id}_b) = (\mathrm{hom}(a,b) \ni g \mapsto \mathrm{id}_b \circ g = g \in \mathrm{hom}(a,b)) = \mathrm{id}_{\mathrm{hom}(a,b)}$$

- composition
For $f\colon b \to c$ and $g : c \to d$ we have

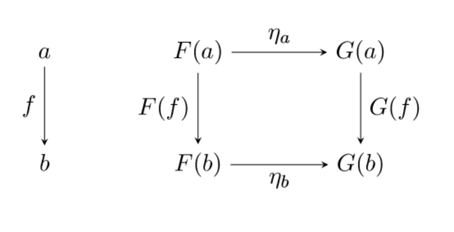
$$\begin{aligned}\mathrm{Hom}(a, g \circ f) &= (\mathrm{hom}(a,b) \ni h \mapsto (g \circ f) \circ h = g \circ (f \circ h) \in \mathrm{hom}(a,d)) \\ &= (\mathrm{hom}(a,c) \ni h \mapsto g \circ h \in \mathrm{hom}(a,d)) \\ &\quad \circ (\mathrm{hom}(a,b) \ni h \mapsto f \circ h \in \mathrm{hom}(a,c)) \\ &= \mathrm{Hom}(a,g) \circ \mathrm{Hom}(a,f).\end{aligned}$$

Definition: Natural Transformations

Let F and G be two functors between two categories C and D. A natural transformation η from F to G is a family of morphisms that satisties:

- η associates to every object a in C a morphism $\eta_a : F(a) \to G(a)$ between objects of D. η_a is called the component of η at a.
- The components of η must be such that for a morphism $f: a \to b$ we have

$$\eta_b \circ F(f) = G(f) \circ \eta_a.$$



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A natural transformation η is a natural isomorphism (or natural equivalence or isomorphism of functors) if for every object $a \in \mathrm{C}$ the component morphism η_a is an isomorphism in D.

Example: Tensor-hom adjunction

Let Ab be the category of abelian groups, i.e. the category whose objects are abelian groups and whose morphisms are group homomorphisms.

For all abelian groups a, b, c, we have a natural isomorphism (a natural transformation that is isomorphic)

$$\mathrm{Hom}(a \otimes b, c) \to \mathrm{Hom}(a, \mathrm{Hom}(b, c)).$$

To be precise, the two functors are

$$\mathrm{Hom}(\cdot \otimes \cdot, \cdot) : \mathrm{Ab}^{\mathrm{op}} \times \mathrm{Ab}^{\mathrm{op}} \times \mathrm{Ab} \to \mathrm{Ab}$$

and

$$\mathrm{Hom}(\cdot, \mathrm{Hom}(\cdot, \cdot)) : \mathrm{Ab}^{\mathrm{op}} \times \mathrm{Ab}^{\mathrm{op}} \times \mathrm{Ab} \to \mathrm{Ab}.$$

Definition: Functor category

Let C be a small category (objects and morphisms form sets) and D an arbitrary category. The category of functors from C to D has

- as objects covariant functors from C to D,
- as morphisms the natural transformations between such functors.

This is a category because natural transformations can be composed.

Let $\eta(a) : F(a) \to G(a)$ be a natural transformation from the functor $F : \mathrm{C} \to \mathrm{D}$ to the functor $G : \mathrm{C} \to \mathrm{D}$ and $\mu(a) : G(a) \to H(a)$ be a natural transformation from the functor G to the functor H, then we have a natural transformation $\mu(a)\eta(a) : F(a) \to H(a)$ from F to H.

Definition: Presheaf over a category

A presheaf over a category C is a functor

$$X : \mathrm{C}^{\mathrm{op}} \to \mathrm{Set}$$

For an object a of C we denote by $X_a = X(a)$ the evaluation of X at a. The set X_a is called the fibre of the presheaf X at a and elements of X_a are called sections of X over a.

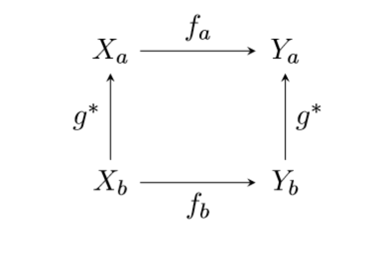
For a morphism $g : a \to b$ in C we write the induced map from X_b to X_a as $g^* = X(g) : X_b \to X_a$.

Definition: Morphism of a Presheaf

Given two presheaves X and Y over C, a morphism of presheaves $f\colon X \to Y$ simply is a natural transformation from X to Y.

In other words, a morphism of presheaves is determined by a collection of maps $f_a : X_a \to Y_a$ such that for any morphism $g\colon a \to b$ in C we have

$$f_a g^* = g^* f_b.$$



Recovering the traditional definition of a presheaf

If C is the poset of open sets in a topological space, interpreted as a category, then we recover the traditional definition of a presheaf on a topological space.

If we want to work with presheaves of Abelian groups on a topological space, then instead of considering functors from C^{op} to the category of sets, we consider functors from C^{op} to the category of Abelian groups.

Category of presheaves

Presheaves over a category C naturally form a category, denoted by $\hat{\mathrm{C}}$.

Objects of the category of presheaves over a category C are exactly the functors

$$X\colon \mathrm{C}^{\mathrm{op}} \to \mathrm{Set}$$

and morphisms are given by natural transformations between functors.

Therefore, this is a first example of a functor category.

Yoneda Embedding

The Yoneda embedding is the functor

$$h : \mathrm{C} \to \hat{\mathrm{C}}$$

whose values at an object a of C is the presheaf

$$h_a = \mathrm{Hom}_{\mathrm{C}}(\cdot, a).$$

Yoneda Lemma

For any presheaf X over a category C, there is a natural bijection of the form

$$\mathrm{hom}_{\hat{\mathrm{C}}}(h_a, X) \overset{\sim}{\to} X_a$$

$$(h_a \overset{f}{\to} X) \mapsto f_a(1_a)$$

Proof: Yoneda Lemma

We only need to define the reverse direction. Given a section s of X over a, i.e. an element $s \in X_a$, we define a collection of morphisms as follows:

$$f_b : \hom_{\mathrm{C}}(b, a) \to X_b$$

For each morphism $g : b \to a$, the section $f_b(g)$ is the element $f_b(g) = g^*(s)$.

This collection defines a morphism $f : h_a \to X$ between the two functors h_a and X since it is a natural transformation between the functors h_a and X in the category C.

Then we just have to verify that $s \mapsto f$ is a two-sided inverse of h_a.

Corollary 1

The yoneda embedding $h : \mathrm{C} \to \hat{\mathrm{C}}$ is a fully faithful functor, i.e. the induced function

$$h_{a,b} : \hom_{\mathrm{C}}(a, b) \to \hom_{\hat{\mathrm{C}}}(h_a, h_b)$$

is bijective.

Proof: Corollary 1

Use the Yoneda Lemma with $X = h_b$. By the Yoneda lemma we have a bijection of the form $(h_b)_a \to \hom_{\widehat{\mathrm{C}}}(h_a, h_b)$.
But

$$h_b = \mathrm{Hom}_{\mathrm{C}}(\cdot, b)$$

and thus

$$(h_b)_a = \hom_{\mathrm{C}}(a, b).$$

Therefore we have a bijection

$$h_{a,b} : \hom_{\mathrm{C}}(a, b) \to \hom_{\hat{\mathrm{C}}}(h_a, h_b)$$

Corollary 2: Uniqueness of representing objects

Since the Yoneda embedding is a full and faithful functor, an isomorphism of representable presheaves (presheaf naturally isomorphic to the contravariant Hom functor)

$$h_a \simeq h_b$$

must come from an isomorphism

$$a \simeq b$$

in the original category C

Proposition

Let C, D be categories. Suppose D admits small colimits. Then each functor $\mathrm{F} : \mathrm{C} \to \mathrm{D}$ factorizes in the following manner

$$\mathrm{C} \xrightarrow{h} \hat{\mathrm{C}} \xrightarrow{\hat{\mathrm{F}}} D$$

where h is the Yoneda embedding and $\hat{\mathrm{F}} : \hat{\mathrm{C}} \to D$ is a colimit-preserving functor called the Yoneda extension of F.