Kan Complexes

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Let us recall some basic notions from Algebraic topology.

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Let X be a topological space.

- We can associate the set $\pi_0(X)$ of path components of X to the topological space X.
- We can associate the fundamental group $\pi_1(X, x)$ of X to the topological space X with a given based point $x \in X$.

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To any space topological space X we can associate an invariant $\pi_{\leq 1}(X)$ called the fundamental groupoid of X.

The fundamental groupoid is the category whose objects are the points of X, where a morphism from a point $x \in X$ to a point $y \in Y$ is given by a homotopy class of continuous paths $p : [0, 1] \to X$ with $p(0) = x$ and $p(1) = y$.

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We recover $\pi_0(X)$ as the set of isomorphism classes of objects of the category $\pi_{\leq 1}(X)$.
Each fundamental group $\pi(X, x)$ can be identified with the automorphism group of the point x as an object of the category $\pi_{\leq}(X)$.

Question: "Can we find a 'category-theoretic' invariant of X which contains information about all the homotopy groups?"

We will answer this question to a satisfying extent when we introduce simplicial sets and in particular the singular simplical set $\mathrm{Sing}_\bullet(X)$ of X.

But to what extent does $\mathrm{Sing}_\bullet(X)$ behave like a category?

To give an answer to this question, we will slowly build up to a notion of ∞-categories.

Informally, we can think of the theory of ∞-categories as an attempt at bringing together categories and homotopy theory into a single framework.

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We will refer to Δ as the simplex category.

Simplicial Object

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In other words, a simplicial set is a presheaf over the simplex category Δ.

Definition: Category of simplicial sets

Since simplicial sets are defined as functors, we have a functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$, which we call the category of simplicial sets and denote by Set_{Δ}.

Since any simplicial set $S_\bullet$ is a functor from $\Delta^{\mathrm{op}} \to \mathrm{Set}$, we will write S_n for the value of the functor $S_\bullet$ on the object $[n] \in \Delta$.

Definition: Face map

Let n be a positive integer. For $0 \leq i \leq n$, we let $\delta^i : [n-1] \to [n]$ denote the unique strictly increasing function whose image does not contain the element i, i.e.

$$\delta^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

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If $C_\bullet$ is a simplicial object of a category C, then we can evaluate $C_\bullet$ on the morphism δ^i to obtain a morphism from C_n to C_{n-1}. We will denote this map by $d_i : C_n \to C_{n-1}$ and we call it the ith face map.

Definition: Degeneracy map

Let n be a positive integer. For $0 \leq i \leq n$, we let $\sigma^i : [n+1] \to [n]$ denote the unique strictly increasing function whose image does not contain the element i, i.e.

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If $C_\bullet$ is a simplicial object of a category C, then we can evaluate $C_\bullet$ on the morphism σ^i to obtain a morphism from C_n to C_{n-1}. We will denote this map by $s_i : C_n \to C_{n-1}$ and we call it the ith degeneracy map.

Definition: Standard n-simplex

For $n \geq 0$, we let Δ^n denote the simplicial set given by

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This is the standard n-simplex and we extend to the case $n = -1$ by setting $\Delta^{-1} = \emptyset$.

The standard n-simplex Δ^n as defined above is indeed a functor from Δ^{op} to Set, since $\mathrm{Hom}_C(\cdot, a) : C \to \mathrm{Set}$ is a contravariant functor for any object $a \in C$.

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By the Yoneda lemma, we have the following universal property for the standard n-simplex Δ^n: For every simplicial set $X_\bullet$, we have a bijection

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X_\bullet) \simeq X_n.$$

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This fact allows us to identify n-simplices of $X_\bullet$ with maps of simplicial sets $\sigma : \Delta^n \to X_\bullet$.

Definition: Simplicial subset of a simplicial set

Let $S_\bullet$ be a simplicial set. Suppose that for every integer $n \geq 0$ we have a subset $T_n \subseteq S_n$ such that the face and degeneracy maps $d_i : S_n \to S_{n-1}$ and $s_i : S_n \to S_{n+1}$ sends T_n into T_{n-1} and T_{n+1}, respectively.

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Then the collection $\{T_n\}_{n\geq 0}$ inherits the structure of a simplicial set $T_\bullet$ and in this case we say that $T_\bullet$ is a simplicial subset of $S_\bullet$ and we can make use of the notation $T_\bullet \subseteq S_\bullet$.

Definition: Boundary of Δ^n

For $n \geq 0$, we define a simplicial set $(\partial\Delta^n) : \Delta^{\mathrm{op}} \to \mathrm{Set}$ by the formula

$$(\partial\Delta^n)([m]) = \{\alpha \in \mathrm{Hom}_{\Delta}([m],[n]) : \alpha \text{ is not surjective}\}.$$

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We can regard $\partial\Delta^n$ as a simplicial subset of the standard n-simplex Δ^n.

We denote by $|\Delta^n|$ the topological simplex of dimension n, i.e.

$$|\Delta^n| := \{(t_0, \ldots, t_n) \in [0,1]^{n+1} : t_0 + t_1 + \ldots + t_n = 1\}.$$

We can also introduce the notion of connected components for simplicial sets.

Definition: Summand of a simplicial set

Let $S_\bullet$ be a simplicial set and let $S'_\bullet \subseteq S_\bullet$ be a simplicial subset of $S_\bullet$. $S'_\bullet$ is a summand of $S_\bullet$ decomposes as a coproduct $S'_\bullet \sqcup S''_\bullet$, for some other simplicial subset $S''_\bullet \subseteq S_\bullet$.

Definition: Connected simplicial set

Let $S_\bullet$ be a simplicial set. $S_\bullet$ is connected it is non-empty and every summand $S'_\bullet \subseteq S_\bullet$ is either empty or coincides with $S_\bullet$.

Definition: Connected Components of a simplicial set

Let $S_\bullet$ be a simplicial set. We will say that a simplicial subset $S'_\bullet \subseteq S_\bullet$ is a connected component of $S_\bullet$ if $S'_\bullet$ is a summand of $S_\bullet$ and $S'_\bullet$ is connected. We denote the set of all connected components of $S_\bullet$ by $\pi_0(S_\bullet)$.

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- To each object $[n] \in \Delta$, we assign the set $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$ of singular n-simplices in X.
- To each non-decreasing map $\alpha : [m] \to [n]$, we assign the map $\mathrm{Sing}_n(X) \to \mathrm{Sing}_m(X)$ given by precomposition with the continuous map

$$\begin{aligned} |\Delta^m| &\to |\Delta^n| \\ (t_0, t_1, \ldots, t_m) &\mapsto \Big(\sum_{\alpha(i)=0} t_i, \sum_{\alpha(i)=1} t_i, \ldots, \sum_{\alpha(i)=n} t_i \Big). \end{aligned}$$

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$\mathrm{Sing}_\bullet(X)$ is the so called singular simplicial set of X.
The above construction yields a functor $X \mapsto \mathrm{Sing}_\bullet(X)$ from the category of topological spaces to the category of simplicial sets, which will be denoted by $\mathrm{Sing}_\bullet : \mathrm{Top} \to \mathrm{Set}_\Delta$.

Let X be a topological space. By definition, n-simplices of the simplicial set $\mathrm{Sing}_\bullet(X)$ are continuous maps $|\Delta^n| \to X$, which yields a bijection

$$\mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{Sing}_\bullet(X)).$$

In order to make use of this observation in a more general setting, we introduce the notion of geometric realization.

Definition: Geometric Realization

Let $S_\bullet$ be a simplicial set and Y a topological space. A map of simplicial sets $u : S_\bullet \to \mathrm{Sing}_\bullet(Y)$ exhibits Y as a geometric realization of $S_\bullet$ if for every topological space X the composite map

$$\mathrm{Hom}_{\mathrm{Top}}(Y, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sing}_\bullet(Y), \mathrm{Sing}_\bullet(X)) \overset{\circ u}{\to} \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, \mathrm{Sing}_\bullet(X))$$

is bijective.

For every $n \geq 0$, the identity map $|\Delta^n| \simeq |\Delta^n|$ determines an n-simplex of the simplicial set $\mathrm{Sing}_\bullet(|\Delta^n|)$, which we can identify with a map of simplicial sets $\Delta^n \to \mathrm{Sing}_\bullet(|\Delta^n|)$ which exhibits $|\Delta^n|$ as a geoemtric realization of Δ^n.

Let $S_\bullet$ be a simplicial set. If there exists a map $u : S_\bullet \to \mathrm{Sing}_\bullet(Y)$ that exhibits Y as a geometric realization of $S_\bullet$, then the topological space Y is determined up to homeomorphism and depends functorially on $S_\bullet$. To emphasize this dependence, we write $|S_\bullet|$ to denote the geometric realization of $S_\bullet$.

Proposition

For every simplicial set $S_\bullet$ there exists a topological space Y and a map $u : S_\bullet \to \mathrm{Sing}_\bullet(Y)$ which exhibits Y as a geometric realization of $S_\bullet$.

We will only provide a sketch of a potential proof of this proposition. The main insight required for a full proof is that every simplicial set can be presented as a colimit of simplices. Then we can make use of the following Lemma

Lemma

Let C be a small category and let $F : C \to \mathrm{Set}_\Delta$ be a functor. Let $S_\bullet = \varinjlim_{c \in C} F(c)_\bullet$ be a colimit of F. If each of the simplicial sets $F(c)_\bullet$ admits a geometric realization $|F(c)_\bullet|$, then $S_\bullet$ also admits a geometric realization, given by the colimit $Y = \varinjlim_{c \in C} |F(c)_\bullet|$.

Definition: The Horn Λ_i^n

Given a pair of integers $0 \leq i \leq n$, we define a simplicial set $\Lambda_i^n : \Delta^{\mathrm{op}} \to \mathrm{Set}$ by the formula

$$(\Lambda_i^n)([m]) = \{\alpha \in \mathrm{Hom}_{\Delta}([m],[n]) : [n] \not\subseteq \alpha([m]) \cup \{i\}\}.$$

We regard Λ_i^n as a simplicial subset of the boundary $\partial\Delta^n \subseteq \Delta^n$. We will refer to Λ_i^n as the *i*th horn in Δ^n. We will say that Λ_i^n is an inner horn if $0 < i < n$, and an outer horn if $i = 0$ or $i = n$.

Roughly speaking, one can think of the horn Λ_i^n as obtained from the n-simplex Δ^n by removing its interior together with the face opposite its ith vertex.

Definition: Kan Complex

Let $S_\bullet$ be a simplicial set. We will say that $S_\bullet$ is a Kan complex if it satisfies the following condition

($\star$) For $n > 0$ and $0 \leq i \leq n$, any map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to a map $\sigma : \Delta^n \to S_\bullet$. Here $\Lambda^n_i \subseteq \Delta^n$ denotes the ith horn.

Example

Let X be a topological space. Then the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is a Kan complex.

Let $\sigma_0 : \Lambda^n_i \to \mathrm{Sing}_\bullet(X)$ be a map of simplicial sets for $n > 0$; we wish to show that σ_0 can be extended to an n-simplex of X.

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Using the geometric realization functor, we can identify σ_0 with a continuous map of topological spaces $f_0 : |\Lambda^n_i| \to X$; we wish to show that f_0 factors as a composition

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We take f to be the composition $f_0 \circ r$, where r is any continuous retraction of $|\Delta^n|$ onto the subset $|\Lambda^n_i|$. A possible candidate is the map r given by the formula

$$r(t_0, \ldots, t_n) = (t_0 - c, \ldots, t_{i-1} - c, t_i + nc, t_{i+1} - c, \ldots, t_n - c),$$

where $c = \min\{t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}$.

Example

Let $G_\bullet$ be a simplicial group (that is, a simplicial object of the category of groups). Then the underlying simplicial set of $G_\bullet$ is a Kan complex.

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There are three major reasons for why Kan complexes play such an important role in the theory of ∞-categories:

1. Every Kan Complex is an ∞-category
2. For any pair X, Y of objects in a ∞-category C, we can associate a Kan Complex $\mathrm{Hom}_C(X, Y)$, which is called the space of maps from X to Y

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1. Every Kan Complex is an ∞-category
2. For any pair X, Y of objects in a ∞-category C, we can associate a Kan Complex $\mathrm{Hom}_C(X, Y)$, which is called the space of maps from X to Y
3. The collection of all Kan complexes can be organized into an ∞-category, which is called the ∞-category of spaces.

Recall that to any topological space X, we can associate the set $\pi_0(X)$ of path components of X and given a base point $x \in X$ we can associate the fundamental group $\pi_1(X)$.

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Combining the set of path components and the fundamental groups $\{\pi_1(X, x)\}_{x \in X}$ yields the fundamental groupoid $\pi_{\leq 1}(X)$, a category whose objects are the points of X with morphisms from a point $x \in X$ to a point $y \in X$ is given by a homotopy class of continuous paths $p : [0, 1] \to X$ satisfying $p(0) = x$ and $p(1) = y$.

We can recover the set of path components $\pi_0(X)$ from the set of isomorphism classes of objects of the category $\pi_{\leq 1}(X)$ and each fundamental group $\pi_1(X, x)$ can be identified with the automorphism group of the point x as an object of the category.

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Despite the importance of the invariant $\pi_{\leq 1}(X)$ of a topological space X, this is far from a complete invariant, since it does not contain any information about higher homotopy groups $\{\pi_n(X, x)\}_{n \geq 2}$.

This naturally raises the question: Is there a "category-theoretic" invariant of a topological space X, in the spirit of the fundamental groupoid $\pi_{\leq 1}(X)$, which contains information about all the homotopy groups of X?

This question is partially answered by what we have discussed up to now. Every topological space X determines a simplicial set $\mathrm{Sing}_\bullet(X)$. The homotopy groups of X can be reconstructed from the simplicial set $\mathrm{Sing}_\bullet(X)$ by a simple combinatorial procedure and we can use this procedure even for Kan Complexes.

We saw that every topological space X determines a simplicial set which is in fact a Kan complex and Kan complexes form a particular class of simplicial sets. On the other hand we can consider a different class of simplicial sets which arise from the theory of categories.

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To each category C we associate a simplicial set $N_\bullet(C)$, called the nerve of C. This construction $C \mapsto N_\bullet(C)$ turns out to be fully faithful, which allows us to consider any category C as a simplicial set.
In particular, one can show that a simplicial set $S_\bullet$ belongs to the essential image of the functor $C \mapsto N_\bullet(C)$ if and only if it satisfies some lifting condition.

Proposition

Let $S_\bullet$ be a simplicial set. Then $S_\bullet$ is isomorphic to the nerve of a category if and only if it satisfies the following condition:

($\star\star$) For every pair of integers $0 < i < n$ and every map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$, there exists a unique map $\sigma : \Delta^n \to S_\bullet$ such that $\sigma_0 = \sigma|_{\Lambda^n_i}$.

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Sometimes the condition ($\star'$) is referred to as the weak Kan extension condition.

Example

Every Kan complex is an ∞-category and in particular, if X is a topological space, then the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is an ∞-category.

Example

For every category C, the nerve $N_\bullet(C)$ is an ∞-category.

We summarize the various classes of simplicial sets we have encountered so far with the following diagram:

$$
\begin{array}{ccccc}
\{\text{Categories}\} & \overset{N_\bullet}{\hookrightarrow} & \{\infty\text{-Categories}\} & \supset & \{\text{Kan Complexes}\} \\
 & & \cap & & \uparrow \scriptstyle{\text{Sing}_\bullet} \\
 & & \{\text{Simplicial Sets}\} & & \{\text{Topological Spaces}\}
\end{array}
$$

Let $C = S_\bullet$ be an ∞-category. An object of C is a vertex of the simplicial set $S_\bullet$, i.e. an element of the set S_0. A morphism of C is an edge of the simplicial set $S_\bullet$.

Example: $N_\bullet(C)$

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- For every object $X \in C$, the identity morphism id_X does not depend on whether we view X as an object of the category C or the ∞-category $N_\bullet(C)$.

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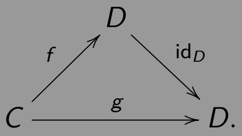
- For every point $x \in X$, the identity morphism id_x is the constant path $[0,1] \to X$ taking the value x.

Homotopies of morphisms

Let C be an ∞-category and let $f, g : C \to D$ be a pair of morphisms in C having the same source and target. A homotopy from f to g is a 2-simplex σ of C satisfying $d_0(\sigma) = id_D$, $d_1(\sigma) = g$, and $d_2(\sigma) = f$, as depited in the diagram



Example

Let C be an ordinary category. Then a pair of morphisms $f, g : C \to D$ in C are homotopic as morphisms of the ∞-category $N_\bullet(C)$ if and only if $f = g$.

Example

Let X be a topological space. Suppose we are given points $x, y \in X$ and a pair of continuous paths $f, g : [0,1] \to X$ satisfying $f(0) = x = g(0)$ and $f(1) = y = g(1)$. Then f and g are homotopic as morphisms of the ∞-category $\mathrm{Sing}_\bullet(X)$ if and only if the paths f and g are homotopic relative to their endpoints, that is, if and only if there exists a continuous function $H : [0,1] \times [0,1] \to X$ satisfying

$$H(s,0) = f(s) \quad H(s,1) = g(s) \quad H(0,t) = x \quad H(1,t) = y.$$

Proposition

Let C be an ∞-category containing objects $X, Y \in C$, and let E denote the collection of all morphisms from X to Y in C. Then homotopy is an equivalence relation on E.

For any morphism $f : X \to Y$ in C, the degenerate 2-simplex $s_1(f)$ is a homotopy from f to itself. It follows that homotopy is a reflexive relation on E.

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We will show that for three morphisms $f, g, h : X \to Y$ from X to Y f homotopic to g and f homotopic to h implies that g is homotopic to h. First, we observe that in the case $f = h$ we get symmetry, so we can replace f homotopic to g with g homotopic to f in above claim. Then we have transitivity and we are done.

So we just need to prove above claim. Let σ_2 and σ_3 be 2-simplices of C which are homotopies from f to h and f to g, respectively, and let σ_0 be the 2-simplex given by the constant map $\Delta^2 \to \Delta^0 \xrightarrow{Y} C$. Then the tuple $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda^3_1 \to C$, depicted informally by the diagram

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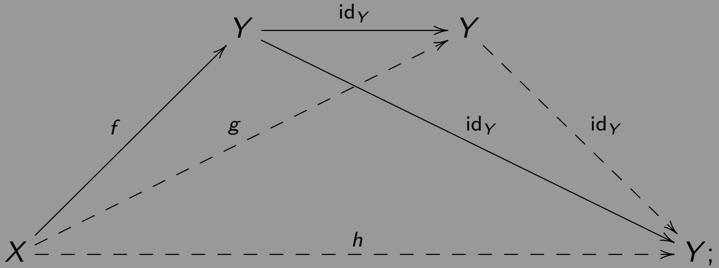


with the dotted arrows representing the boundary of the "missing" face of the horn Λ^3_1. Our hypothesis that C is an ∞-category guarantees that τ_0 can be extended to a 3-simplex τ of C. We can then regard the face $d_1(\tau)$ as a homotopy from g to h.

Let C be an ∞-category. Suppose we are given objects $X, Y, Z \in C$ and morphisms $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$. We will say that h is a composition of f and g if there exists a 2-simplex σ of C satisfying $d_0(\sigma) = g$, $d_1(\sigma) = h$ and $d_2(\sigma) = f$.

Proposition

Let C be an ∞-category containing morphisms $f : X \to Y$ and $g : Y \to Z$. Then:

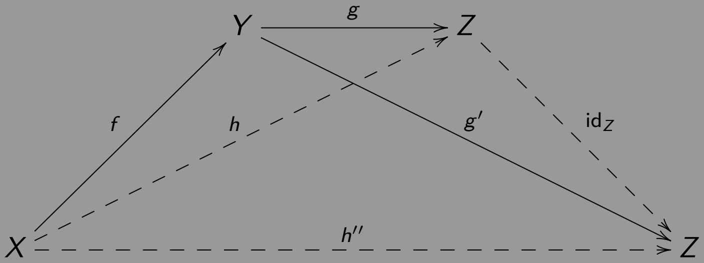
1. There exists a morphism $h : X \to Z$ which is a composition of f and g.
2. Let $h : X \to Z$ be a composition of f and g, and let $h' : X \to Z$ be another morphism in C having the same source and target. Then h' is a composition of f and g if and only if h' is homotopic to h.

Proposition

Let C be an ∞-category. Suppose we are given a pair of homotopic morphisms $f, f' : X \to Y$ in C and a pair of homotopic morphisms $g, g' : Y \to Z$ in C. Let h be a composition of f and g, and let h' be a composition of f' and g'. Then h is homotopic to h'.

Let h'' be a composition of f and g'. Since homotopy is an equivalence relation, it will suffice to show that both h and h' are homotopic to h''. We will show that h is homotopic to h'' since the proof that h' is homotopic to h'' is simlar. Let σ_3 be a 2-simplex of C which witnesses h as a composition of f and g, let σ_2 be a 2-simplex of C which witnesses h as a composition of f and g, let σ_2 be the 2-simplex of C which witnesses h'' as a composition of f and g', and let σ_0 be a 2-simplex of C which is a homotopy from g to g'.

Then the tuple $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda^3_1 \to C$ which we depict imformally as a diagram



where the dotted arrows indicate the boundary of the "missing" face of the horn Λ^3_1. Using our assumption that C is an ∞-category, we can extend τ_0 to a 3-simplex τ of C. Then the face $d_1(\tau)$ is a homotopy from h to h''.

Let C be an ∞-category. For every pair of objects $X, Y \in C$, we let $\mathrm{Hom}_{\mathrm{hC}}(X, Y)$ denote the set of homotopy classes of morphisms from X to Y in C.

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It follows from the 2 previous propositions that for every triple of objects $X, Y, Z \in C$, there is a unique composition law

$$\circ : \mathrm{Hom}_{\mathrm{hC}}(Y, Z) \times \mathrm{Hom}_{\mathrm{hC}}(X, Y) \to \mathrm{Hom}_{\mathrm{hC}}(X, Z)$$

satisfying the identity $[g] \circ [f] = [h]$ whenever $h : X \to Z$ is a composition of f and g in the ∞-category C.

Proposition

Let C be an ∞-category. Then

1. The composition law above is associative, that is for every triple of composable morphisms $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ in C, we have an equality $([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f])$ in $\mathrm{Hom}_{\mathrm{hC}}(W, Z)$.
2. For every object $X \in C$, the homotopy class $[\mathrm{id}_X] \in \mathrm{Hom}_{\mathrm{hC}}(X, X)$ is a two-sided identity with respect to the composition law above. That is, for every morphism $f : W \to X$ in C and every morphism $g : X \to Y$ in C, we have $[\mathrm{id}_X] \circ [f] = [f]$ and $[g] \circ [\mathrm{id}_X] = [g]$.

The Homotopy Category

Let C be an ∞-category. We define a category hC as follows:

- The objects of hC are the objects of C.
- For every pair of objects $X, Y \in C$, we let $\mathrm{Hom}_{\mathrm{hC}}(X, Y)$ denote the collection of homotopy classes of morphisms from X to Y in the ∞-category C (as discussed just above).
- For every object $X \in C$, the identity morphism from X to itself in hC is given by the homotopy class $[\mathrm{id}_X]$.
- Composition of morphisms is defined as above.

We will refer to hC as the homotopy category of the ∞-category C.

Example

Let C be an ordinary category. Then the homotopy category of the ∞-category $N_\bullet(C)$ can be identified with C. For instance, the homotopy category $h\Delta^n$ can be identified with $[n] = \{0 < 1 < \ldots < n\}$.

Example

Let X be a topological space, and consider the singular simplicial set $\mathrm{Sing}_\bullet(X)$ as an ∞-category. Then the homotopy category $\mathrm{hSing}_\bullet(X)$ can be identified with the fundamental groupoid $\pi_{\leq 1}(X)$.

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In fact, we can regard the treatment of ∞-categories up to now when restricted to ∞-categories of the form $\mathrm{Sing}_\bullet(X)$ as providing a construction of the fundamental groupoid of X.

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In other words, Let X and Y be simplicial sets, and suppose we are given a pair of morphisms $f_0, f_1 : X \to Y$. A homotopy from f_0 to f_1 is a morphism $h : \Delta^1 \times X \to Y$ satisfying $f_0 = h|_{\{0\}\times X}$ and $f_1 = h|_{\{1\}\times X}$.

The Homotopy Category of Kan Complexes

We define a category hKan as follows:

- The objects of hKan are Kan complexes.
- If X and Y are Kan complexes, then $\mathrm{Hom}_{\mathrm{hKan}}(X, Y) = [X, Y] = \pi_0(\mathrm{Fun}(X, Y))$ is the set of homotopy classes of morphisms from X to Y.
- If X, Y, Z are Kan complexes, then the composition law

$$\circ : \mathrm{Hom}_{\mathrm{hKan}}(Y, Z) \times \mathrm{Hom}_{\mathrm{hKan}}(X, Y) \to \mathrm{Hom}_{\mathrm{hKan}}(X, Z)$$

is characterized by the formula $[g] \circ [f] = [g \circ f]$.

We will refer to hKan as the Homotopy category of Kan Complexes.

Let $f : X \to Y$ be a morphism of simplicial sets. We will say that a morphism $g : Y \to X$ is a homotopy inverse to f if the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity morphisms id_X and id_Y respectively.
We say that $f : X \to Y$ is a homotopy equivalence if it admits a homotopy inverse g.

Let $f : X \to Y$ be a homotopy equivalence of topological spaces. Then the induced map of singular simplicial sets $\mathrm{Sing}_\bullet(f) : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(Y)$ is a homotopy equivalence.

Pointed simplicial set

A pointed simplicial set is a pair (X, x) where X is a simplicial set and x is a vertex of X. If X is a Kan complex, then we refer to the pair (X, x) as a pointed Kan complex.
A pointed map between two Kan Complexes $(X, x), (Y, y)$ is a morphism of Kan complexes $f : X \to Y$ satisfying $f(x) = y$. We let Kan_* denote the category whose objects are pointed Kan complexes and whose morphisms are pointed maps.

Pointed homotopic

Let (X, x) and (Y, y) be simplicial sets, and suppose we are given a pair of pointed maps $f, g : X \to Y$, which we identify with vertices of the simplicial set $\mathrm{Fun}(X, Y) \times_{\mathrm{Fun}(\{x\}, Y)} \{y\}$.
We say that f and g are pointed homotopic if they belong to the same connected component of $\mathrm{Fun}(X, Y) \times_{\mathrm{Fun}(\{x\}, Y)} \{y\}$.

Homotopy Category of pointed Kan complexes

We define a category hKan_* as follows:

- The objects of hKan_* are pointed Kan complexes (X, x).
- If (X, x) and (Y, y) are pointed Kan complexes, then $\mathrm{Hom}_{\mathrm{hKan}_*}((X, x), (Y, y)) = [X, Y]_*$ is the set of pointed homotopy classes of morphisms from (X, x) to (Y, y).
- If $(X, x), (Y, y), (Z, z)$ are pointed Kan complexes, then the composition law

  $$\circ : \mathrm{Hom}_{\mathrm{hKan}_*}((Y, y), (Z, z)) \times \mathrm{Hom}_{\mathrm{hKan}_*}((X, x), (Y, y)) \to \mathrm{Hom}_{\mathrm{hKan}_*}((X, x), (Z, z))$$

  is characterized by the formula $[g] \circ [f]g = [g \circ f]$.

We refer to hKan_* as the homotopy category of pointed Kan complexes.

We are now ready to construct the homotopy groups of Kan complexes

Let X be a topological space and let $x \in X$ be a point.

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For every positive integer n, we let $\pi_n(X, x)$ denote the set of homotopy classes of pointed maps $(S^n, x_0) \to (X, x)$ where S^n denotes a sphere of dimension n and $x_0 \in S^n$ is a chosen base point.

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The set $\pi_n(X, x)$ can be endowed with the structure of a group, which we refer to as the nth homotopy group of X with respect to the base point x.

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We can therefore identify pointed maps $(S^n, x_0) \to (X, x)$ with maps of simplicial sets $f : \Delta^n \to \mathrm{Sing}_\bullet(X)$ which carry the boundary $\partial\Delta^n$ to the simplicial subset $\{x\} \subseteq \mathrm{Sing}_\bullet(X)$. This gives us a direct construction of the homotopy group $\pi_n(X, x)$ in terms of the simplicial set $\mathrm{Sing}_\bullet(X)$.

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This construction can be applied directly to any Kan complex.

Whitehead's Theorem for Kan complexes

Let $f : X \to Y$ be a morphism of Kan complexes. Then f is a homotopy equivalence if and only if it satisfies the following two conditions:

- The map of sets $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.
- For every vertex $x \in X$ having image $y = f(x)$ in Y and every positive integer n, the map of homotopy groups $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, y)$ is bijective.

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The main annoyance is that f being a homotopy equivalence guarantees that we have an induced isomorphism in the homotopy category hKan of Kan complexes. But Homotopy groups of X and Y are computed by viewing (X, x) and (Y, y) as objects of hKan_*.

Milnor's Theorem

The geometric realization functor $|\bullet| : \mathrm{Set}_{\Delta} \to \mathrm{Top}$ induces an equivalence from the homotopy category hKan to the full subcategory of hTop spanned by those topological spaces X which have the homotopy type of a CW complex.

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1. Show that geometric realization is well-defined at the level of homotopy categories
2. Show that the geometric realization functor $|\bullet| : \mathrm{hKan} \to \mathrm{hTop}$ is fully faithful.
3. Show that if Y is a topological space, then the counit map $v_Y : |\mathrm{Sing}_\bullet(Y)| \to Y$ is a homotopy equivalence if and only if Y has the homotopy type of a CW complex.