Kan Complexes and their relevance in the theory of ∞-categories

Algebraic and Differential Topology seminar

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Contents

1 Simplicial Sets

§1.1 Simplex Category

Definition 1.1.1. We define a category Δ as follows:

- The objects of Δ are linearly ordered sets of the form $[n]$ for $n \geq 0$.
- A morphism from $[m]$ to n in the category Δ is a function $\alpha : [m] \to [n]$ which is nondecreasing: that is, for each $0 \leq i \leq j \leq m$, we have $0 \leq \alpha(i) \leq \alpha(j) \leq n$.

We will refer to Δ as the simplex category.

§1.2 Simplicial object

Definition 1.2.1. Let C be any category. A simplicial object of C is a functor $\Delta^{\text{op}} \to C$. Dually, a cosimplicial object of C is a functor $\Delta \to C$.

§1.3 Simplicial set

Definition 1.3.1. A simplicial set is a functor $\Delta^{\text{op}} \to \text{Set}$ from the simplex category to the category of sets.

In other words, a simplicial set is a presheaf over the simplex category Δ.

§1.4 Category of simplicial sets

Definition 1.4.1. Since simplicial sets are defined as functors, we have a functor category $\text{Fun}(\Delta^{\text{op}}, \text{Set})$, which we call the category of simplicial sets and denote by Set_Δ.

Remark 1.4.2 — Since any simplicial set $S_\bullet$ is a functor from $\Delta^{\text{op}} \to \text{Set}$, we will write S_n for the value of the functor $S_\bullet$ on the object $[n] \in \Delta$.

§1.5 Standard n-simplex

Definition 1.5.1. For $n \geq 0$, we let Δ^n denote the simplicial set given by

$$([m] \in \Delta) \mapsto \text{Hom}_\Delta([m], [n]).$$

This is the standard n-simplex and we extend to the case $n = -1$ by setting $\Delta^{-1} = \emptyset$.

Definition 1.5.2 (Simplicial subset of a simplicial set)**.** Let $S_\bullet$ be a simplicial set. Suppose that for every integer $n \geq 0$ we have a susbet $T_n \subseteq S_n$ such that the face and degeneracy maps $d_i : S_n \to S_{n-1}$ and $s_i : S_n \to S_{n+1}$ sends T_n into T_{n-1} and T_{n+1}, respectively. Then the collection $\{T_n\}_{n\geq 0}$ inherits the structure of a simplicial set $T_\bullet$ and in this case we say that $T_\bullet$ is a simplicial subset of $S_\bullet$ and we can make use of the notation $T_\bullet \subseteq S_\bullet$.

Definition 1.5.3 (Boundary of Δ^n)**.** For $n \geq 0$, we define a simplicial set $(\partial\Delta^n) : \Delta^{\mathrm{op}} \to \mathrm{Set}$ by the formula

$$(\partial\Delta^n)([m]) = \{\alpha \in \mathrm{Hom}_\Delta([m],[n]) : \alpha \text{ is not surjective}\}.$$

We can regard $\partial\Delta^n$ as a simplicial subset of the standard n-simplex Δ^n.

Remark 1.5.4 — Δ^n as defined above is indeed a functor from Δ^{op} to Set, since $\mathrm{Hom}_C(\cdot, a) : C \to \mathrm{Set}$ is a contravariant functor for any object $a \in C$.

By the Yoneda lemma, we have the following universal property for the standard n-simplex Δ^n: For every simplicial set $X_\bullet$, we have a bijection

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X_\bullet) \simeq X_n.$$

This fact allows us to identify n-simplices of $X_\bullet$ with maps of simplicial sets $\sigma : \Delta^n \to X_\bullet$.

Remark 1.5.5 — We denote by $|\Delta^n|$ the topological simplex of dimension n, i.e.

$$|\Delta^n| := \{(t_0, \ldots, t_n) \in [0,1]^{n+1} : t_0 + t_1 + \ldots + t_n = 1\}.$$

§1.6 Connectedness of simplicial sets

We can also introduce the notion of connected components for simplicial sets.

Definition 1.6.1. Let $S_\bullet$ be a simplicial set and let $S'_\bullet \subseteq S_\bullet$ be a simplicial subset of $S_\bullet$. $S'_\bullet$ is a summand of $S_\bullet$ decomposes as a coproduct $S'_\bullet \sqcup S''_\bullet$, for some other simplicial subset $S''_\bullet \subseteq S_\bullet$.

Definition 1.6.2. Let $S_\bullet$ be a simplicial set. $S_\bullet$ is connected it is non-empty and every summand $S'_\bullet \subseteq S_\bullet$ is either empty or coincides with $S_\bullet$.

Definition 1.6.3. Let $S_\bullet$ be a simplicial set. We will say that a simplicial subset $S'_\bullet \subseteq S_\bullet$ is a connected component of $S_\bullet$ if $S'_\bullet$ is a summand of $S_\bullet$ and $S'_\bullet$ is connected. We denote the set of all connected components of $S_\bullet$ by $\pi_0(S_\bullet)$.

§1.7 Simplicial set of a Topological space

Definition 1.7.1. Let X be a topological space. We define a simplicial set $\mathrm{Sing}_\bullet(X)$ as follows:

- To each object $[n] \in \Delta$, we assign the set $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$ of singular n-simplices in X.
- To each non-decreasing map $\alpha : [m] \to [n]$, we assign the map $\mathrm{Sing}_n(X) \to \mathrm{Sing}_m(X)$ given by precomposition with the continuous map

$$\begin{aligned} |\Delta^m| &\to |\Delta^n| \\ (t_0, t_1, \ldots, t_m) &\mapsto \Big(\sum_{\alpha(i)=0} t_i, \sum_{\alpha(i)=1} t_i, \ldots, \sum_{\alpha(i)=n} t_i \Big). \end{aligned}$$

$\mathrm{Sing}_\bullet(X)$ is the so called singular simplicial set of X. The above construction yields a functor $X \mapsto \mathrm{Sing}_\bullet(X)$ from the category of topological spaces to the category of simplicial sets, which will be denoted by $\mathrm{Sing}_\bullet : \mathrm{Top} \to \mathrm{Set}_\Delta$.

Let X be a topological space. By definition, n-simplices of the simplicial set $\mathrm{Sing}_\bullet(X)$ are continuous maps $|\Delta^n| \to X$, which yields a bijection

$$\mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{Sing}_\bullet(X)).$$

In order to make use of this observation in a more general setting, we introduce the notion of geometric realization.

§1.8 Geometric Realization

Definition 1.8.1. Let $S_\bullet$ be a simplicial set and Y a topological space. A map of simplicial sets $u : S_\bullet \to \mathrm{Sing}_\bullet(Y)$ exhibits Y as a geometric realization of $S_\bullet$ if for every topological space X the composite map

$$\mathrm{Hom}_{\mathrm{Top}}(Y, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sing}_\bullet(Y), \mathrm{Sing}_\bullet(X)) \overset{\circ u}{\to} \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, \mathrm{Sing}_\bullet(X))$$

is bijective.

Example 1.8.2

For every $n \geq 0$, the identity map $|\Delta^n| \simeq |\Delta^n|$ determines an n-simplex of the simplicial set $\mathrm{Sing}_\bullet(|\Delta^n|)$, which we can identify with a map of simplicial sets $\Delta^n \to \mathrm{Sing}_\bullet(|\Delta^n|)$ which exhibits $|\Delta^n|$ as a geoemtric realizatino of Δ^n.

Remark 1.8.3 — Let $S_\bullet$ be a simplicial set. If there exists a map $u : S_\bullet \to \mathrm{Sing}_\bullet(Y)$ that exhibits Y as a geometric realization of $S_\bullet$, then the topological space Y is determined up to homeomorphism and depends functorially on $S_\bullet$. To emphasize this dependence, we write $|S_\bullet|$ to denote the geometric realization of $S_\bullet$.

Proposition 1.8.4

For every simplicial set $S_\bullet$ there exists a topological space Y and a map $u : S_\bullet \to \mathrm{Sing}_\bullet(Y)$ which exhibits Y as a geometric realization of $S_\bullet$.

Proof. We will only provide a sketch of a potential proof of this proposition. The main insight required for a full proof is that every simplicial set can be presented as a colimit of simplices. Then we can make use of the following Lemma

Lemma 1.8.5

Let C be a small category and let $F : C \to \mathrm{Set}_\Delta$ be a functor. Let $S_\bullet = \varinjlim_{c \in C} F(c)_\bullet$ be a colimit of F. If each of the simplicial sets $F(c)_\bullet$ admits a geometric realization $|F(c)_\bullet|$, then $S_\bullet$ also admits a geometric realization, given by the colimit $Y = \varinjlim_{c \in C} |F(c)_\bullet|$.

■

2 Kan Complexes

§2.1 Definition

Definition 2.1.1 (The Horn Λ^n_i)**.** Given a pair of integers $0 \le i \le n$, we define a simplicial set $\Lambda^n_i : \Delta^{\mathrm{op}} \to \mathrm{Set}$ by the formula

$$(\Lambda^n_i)([m]) = \{\alpha \in \mathrm{Hom}_\Delta([m],[n]) : [n] \not\subseteq \alpha([m]) \cup \{i\}\}.$$

We regard Λ^n_i as a simplicial subset of the boundary $\partial\Delta^n \subseteq \Delta^n$. We will refer to Λ^n_i as the ith horn in Δ^n. We will say that Λ^n_i is an inner horn if $0 < i < n$, and an outer horn if $i = 0$ or $i = n$.

Definition 2.1.2 (Kan Complex)**.** Let $S_\bullet$ be a simplicial set. We will say that $S_\bullet$ is a Kan complex if it satisfies the following condition

($\star$) For $n > 0$ and $0 \le i \le n$, any map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to a map $\sigma : \Delta^n \to S_\bullet$. Here $\Lambda^n_i \subseteq \Delta^n$ denotes the ith horn.

§2.2 Examples

Proposition 2.2.1

Let X be a topological space. Then the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is a Kan complex.

Proof. Let $\sigma_0 : \Lambda^n_i \to \mathrm{Sing}_\bullet(X)$ be a map of simplicial sets for $n > 0$; we wish to show that σ_0 can be extended to an n-simplex of X. Using the geometric realization functor, we can identify σ_0 with a continuous map of topological spaces $f_0 : |\Lambda^n_i| \to X$; we wish to show that f_0 factors as a composition

$$|\Lambda^n_i| \to |\Delta^n| \xrightarrow{f} X.$$

We can identify $|\Lambda^n_i|$ with the subset

$$\{(t_0, \dots, t_n) \in |\Delta^n| : t_j = 0 \text{ for some } j \ne i\} \subseteq |\Delta^n|.$$

We take f to be the composition $f_0 \circ r$, where r is any continuous retraction of $|\Delta^n|$ onto the subset $|\Lambda^n_i|$. A possible candidate is the map r given by the formula

$$r(t_0, \dots, t_n) = (t_0 - c, \dots, t_{i-1} - c, t_i + nc, t_{i+1} - c, \dots, t_n - c),$$

where $c = \min\{t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_n\}$. ∎

Proposition 2.2.2

Let $G_\bullet$ be a simplicial group (that is, a simplicial object of the category of groups). Then the underlying simplicial set of $G_\bullet$ is a Kan complex.

Proof. Let n be a positive integer and $\vec{\sigma} : \Lambda^n_i \to G_\bullet$ be a map of simplicial sets for some $0 \leq i \leq n$, which we will identify with a tuple $(\sigma_0, \sigma_1, \dots, \sigma_{i-1}, \bullet, \sigma_{i+1}, \dots, \sigma_n)$ of elements of the group G_{n-1}. We wish to prove that there exists an element $\tau \in G_n$ satisfying $d_j\tau = \sigma_j$ for $j \neq i$. Let e denote the identity element of G_{n-1}. We first treat the special case where $\sigma_{i+1} = \dots = \sigma_n = e$. If, in addition, we have $\sigma_0 = \sigma_1 = \dots = \sigma_{i-1} = e$, then we can take τ to be the identity element of G_n. Otherwise, there exists soe smallest integer $j < i$ such that $\sigma_j \neq e$. We proceed by descending induction on j. Set $\tau'' = s_j\sigma_j \in G_n$, and consider the map $\vec{\sigma}' : \Lambda^n_i \to G_\bullet$ given by the tuple $(\sigma'_0, \sigma'_1, \dots, \sigma'_{i-1}, \bullet, \sigma'_{i+1}, \dots, \sigma'_n)$ with $\sigma'_k = \sigma_k(d_k\tau'')^{-1}$. Then we have $\sigma'_0 = \sigma'_1 = \dots = \sigma'_j = e$ and $\sigma'_{i+1} = \dots = \sigma'_n = e$. Invoking our inductive hypothesis we conclude that there exists an element $\tau' \in G_n$ satisfying $d_k\tau' = \sigma'_k$ for $k \neq i$. We can then complete the proof by taking τ to be the product $\tau'\tau''$.

If not all of the equalities $\sigma_{i+1} = \dots = \sigma_n = e$ hold, then there exists some largest integer $j > i$ such that $\sigma_j \neq e$. We now proceed by ascending induction on j. Set $\tau'' = s_{j-1}\sigma_j$ and let $\vec{\sigma}' : \Lambda^n_i \to G_\bullet$ be the map given by the tuple $(\sigma'_0, \sigma'_1, \dots, \sigma'_{i-1}, \bullet, \sigma'_{i+1}, \dots, \sigma'_n)$ with $\sigma'_k = \sigma_k(d_k\tau'')^{-1}$ as before. We then have $\sigma_j = \sigma_{j+1} = \dots = \sigma_n = e$, so the inductive hypothesis guarantees the existence of an element $\tau' \in G_n$ satisfying $d_k\tau' = \sigma'_k$ for $k \neq i$. As before, we complete the proof by setting $\tau = \tau'\tau''$. ■

§2.3 Relevance of Kan Complexes

What makes Kan complexes special and worth consideration? To get a glimpse of the power of Kan Complexes, we will give a short introduction to ∞-categories.

Recall that Kan Complexes are a special class of simplicial sets, namely simplicial sets X with the property that for $n > 0$ and $0 \leq i \leq n$, any morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to X$ can be extended to an n-simplex of X. There are three major reasons for why Kan complexes play such an important role in the theory of ∞-categories:

1. Every Kan Complex is an ∞-category
2. For any pair X, Y of objects in a ∞-category C, we can associate a Kan Complex $\mathrm{Hom}_C(X, Y)$, which is called the space of maps from X to Y
3. The collection of all Kan complexes can be organized into an ∞-category, which is called the ∞-category of spaces.

3 ∞-categories

§3.1 Motivation

Recall that to any topological space X, we can associate the set $\pi_0(X)$ of path components of X and given a base point $x \in X$ we can associate the fundamental group $\pi_1(X)$. Combining the set of path components and the fundamental groups $\{\pi_1(X,x)\}_{x\in X}$ yields the fundamental groupoid $\pi_{\leq 1}(X)$, a category whose objects are the points of X with morphisms from a point $x \in X$ to a point $y \in X$ is given by a homotopy class of continuous paths $p : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$.

We can recover the set of path components $\pi_0(X)$ from the set of isomorphism classes of objects of the category $\pi_{\leq 1}(X)$ and each fundamental group $\pi_1(X,x)$ can be identified with the automorphism group of the point x as an object of the category.

Despite the importance of the invariant $\pi_{\leq 1}(X)$ of a topological space X, this is far from a complete invariant, since it does not contain any information about higher homotopy groups $\{\pi_n(X,x)\}_{n\geq 2}$.

This naturally raises the question: Is there a "category-theoretic" invariant of a topological space X, in the spirit of the fundamental groupoid $\pi_{\leq 1}(X)$, which contains information about all the homotopy groups of X?

This question is partially answered by what we have discussed up to now. Every topological space X determines a simplicial set $\mathrm{Sing}_\bullet(X)$. The homotopy groups of X can be reconstructed from the simplicial set $\mathrm{Sing}_\bullet(X)$ by a simple combinatorial procedure. In fact, the same combinatorial procedure can be applied to all Kan Complexes as we shall discover later.

We saw that every topological space X determines a simplicial set which is in fact a Kan complex and Kan complexes form a particular class of simplicial sets. On the other hand we can consider a different class of simplicial sets which arise from the theory of categories.

To each category C we associate a simplicial set $N_\bullet(C)$, called the nerve of C. This construction $C \mapsto N_\bullet(C)$ turns out to be fully faithful, which allows us to consider any category C as a simplicial set.

In particular, one can show that a simplicial set $S_\bullet$ belongs to the essential image of the functor $C \mapsto N_\bullet(C)$ if and only if it satisfies some lifting condition.

Proposition 3.1.1

Let $S_\bullet$ be a simplicial set. Then $S_\bullet$ is isomorphic to the nerve of a category if and only if it satisfies the following condition:

($\star\star$) For every pair of integers $0 < i < n$ and every map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$, there exists a unique map $\sigma : \Delta^n \to S_\bullet$ such that $\sigma_0 = \sigma|_{\Lambda^n_i}$.

So we have two different classes of simplicial sets, which are defined using the following conditions:

($\star$) For $n > 0$ and $0 \leq i \leq n$, any map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to a map $\sigma : \Delta^n \to S_\bullet$. Here $\Lambda^n_i \subseteq \Delta^n$ denotes the ith horn.

(⋆⋆) For every pair of integers $0 < i < n$ and every map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$, there exists a unique map $\sigma : \Delta^n \to S_\bullet$ such that $\sigma_0 = \sigma|_{\Lambda^n_i}$.

It turns out that neither one of these two conditions is a generalization of the other one. But they admit a shared generalization:

§3.2 Definition

Definition 3.2.1. An ∞-category is a simplicial set $S_\bullet$ which satisfies the following condition:

($\star'$) For $0 < i < n$, every map of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to a map $\sigma : \Delta^n \to S_\bullet$.

Sometimes the condition ($\star'$) is referred to as the weak Kan extension condition.

§3.3 Examples

Example 3.3.1

Every Kan complex is an ∞-category and in particular, if X is a topological space, then the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is an ∞-category.

Example 3.3.2

For every category C, the nerve $N_\bullet(C)$ is an ∞-category.

We summarize the various classes of simplicial sets we have encountered so far with the following diagram:

$$\begin{array}{ccccc} \{\text{Categories}\} & \xhookrightarrow{N_\bullet} & \{\infty\text{-Categories}\} & \supset & \{\text{Kan Complexes}\} \\ & & \cap & & \uparrow \mathrm{Sing}_\bullet \\ & & \{\text{Simplicial Sets}\} & & \{\text{Topological Spaces}\} \end{array}$$

Definition 3.3.3. Let $C = S_\bullet$ be an ∞-category. An object of C is a vertex of the simplicial set $S_\bullet$, i.e. an element of the set S_0. A morphism of C is an edge of the simplicial set $S_\bullet$.

Example 3.3.4

Let C be a category and regard the simplicial set $N_\bullet(C)$ as an ∞-category. Then:

- The objects of the ∞-category $N_\bullet(C)$ are the objects of C.
- The morphisms of the ∞-category $N_\bullet(C)$ are the morphisms of C and the source and target of a morphism of C coincide with the source and target of the corresponding morphism of $N_\bullet(C)$
- For every object $X \in C$, the identity morphism id_X does not depend on whether we view X as an object of the category C or the ∞-category $N_\bullet(C)$.

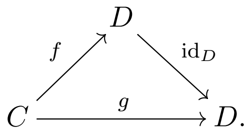
Example 3.3.5

Let X be a topological space, and regard the simplicial set $\text{Sing}_\bullet(X)$ as an ∞-category. Then:

- The objects of the ∞-category $\text{Sing}_\bullet(X)$ are the points of X.
- The morphisms of the ∞-category $\text{Sing}_\bullet(X)$ are continuous paths $f : [0,1] \to X$. The source of a morphism f is the point $f(0)$ and the target is the point $f(1)$.
- For every point $x \in X$, the identity morphism id_x is the constant path $[0,1] \to X$ taking the value x.

§3.4 Homotopies in ∞-categories

Definition 3.4.1 (Homotopies of morphisms)**.** Let C be an ∞-category and let $f, g : C \to D$ be a pair of morphisms in C having the same source and target. A homotopy from f to g is a 2-simplex σ of C satisfying $d_0(\sigma) = id_D$, $d_1(\sigma) = g$, and $d_2(\sigma) = f$, as depited in the diagram



Example 3.4.2

Let C be an ordinary category. Then a pair of morphisms $f, g : C \to D$ in C are homotopic as morphisms of the ∞-category $N_\bullet(C)$ if and only if $f = g$.

Example 3.4.3

Let X be a topological space. Suppose we are given points $x, y \in X$ and a pair of continuous paths $f, g : [0,1] \to X$ satisfying $f(0) = x = g(0)$ and $f(1) = y = g(1)$. Then f and g are homotopic as morphisms of the ∞-category $\text{Sing}_\bullet(X)$ if and only if the paths f and g are homotopic relative to their endpoints, that is, if and only if there exists a continuous function $H : [0,1] \times [0,1] \to X$ satisfying

$$H(s,0) = f(s) \quad H(s,1) = g(s) \quad H(0,t) = x \quad H(1,t) = y.$$

Proposition 3.4.4

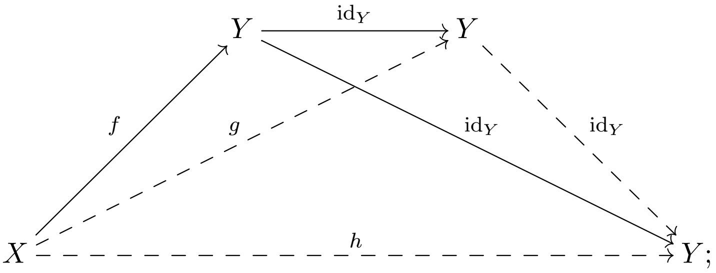
Let C be an ∞-category containing objects $X, Y \in C$, and let E denote the collection of all morphisms from X to Y in C. Then homotopy is an equivalence relation on E.

Proof. For any morphism $f : X \to Y$ in C, the degenerate 2-simplex $s_1(f)$ is a homotopy from f to itself. It follows that homotopy is a reflexive relation on E.

We will show that for three morphisms $f, g, h : X \to Y$ from X to Y f homotopic to g and f homotopic to h implies that g is homotopic to h.

First, we observe that in the case $f = h$ we get symmetry, so we can replace f homotopic to g with g homotopic to f in above claim. Then we have transitivity and we are done.

So we just need to prove above claim. Let σ_2 and σ_3 be 2-simplices of C which are homotopies from f to h and f to g, respectively, and let σ_0 be the 2-simplex given by the constant map $\Delta^2 \to \Delta^0 \xrightarrow{Y} C$. Then the tuple $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda_1^3 \to C$, depicted informally by the diagram



with the dotted arrows representing the boundary of the "missing" face of the horn Λ_1^3. Our hypothesis that C is an ∞-category guarantees that τ_0 can be extended to a 3-simplex τ of C. We can then regard the face $d_1(\tau)$ as a homotopy from g to h. ■

§3.5 Composition of morphisms

Definition 3.5.1. Let C be an ∞-category. Suppose we are given objects $X, Y, Z \in C$ and morphisms $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$. We will say that h is a composition of f and g if there exists a 2-simplex σ of C satisfying $d_0(\sigma) = g$, $d_1(\sigma) = h$ and $d_2(\sigma) = f$.

Proposition 3.5.2

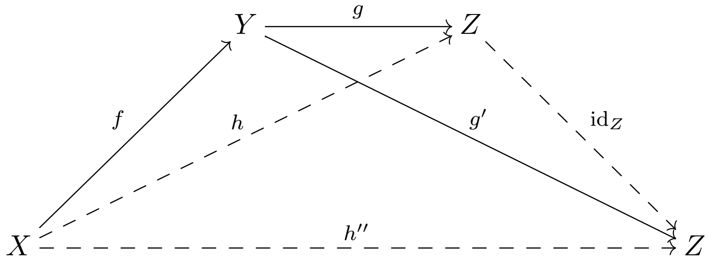
Let C be an ∞-category containing morphisms $f : X \to Y$ and $g : Y \to Z$. Then:

1. There exists a morphism $h : X \to Z$ which is a composition of f and g.
2. Let $h : X \to Z$ be a composition of f and g, and let $h' : X \to Z$ be another morphism in C having the same source and target. Then h' is a composition of f and g if and only if h' is homotopic to h.

Proposition 3.5.3

Let C be an ∞-category. Suppose we are given a pair of homotopic morphisms $f, f' : X \to Y$ in C and a pair of homotopic morphisms $g, g' : Y \to Z$ in C. Let h be a composition of f and g, and let h' be a composition of f' and g'. Then h is homotopic to h'.

Proof. Let h'' be a composition of f and g'. Since homotopy is an equivalence relation, it will suffice to show that both h and h' are homotopic to h''. We will show that h is homotopic to h'' since the proof that h' is homotopic to h'' is simlar. Let σ_3 be a 2-simplex of C which witnesses h as a composition of f and g, let σ_2 be a 2-simplex of C which witnesses h as a composition of f and g, let σ_2 be the 2-simplex of C which witnesses h'' as a composition of f and g', and let σ_0 be a 2-simplex of C which is a homotopy from g to g'. Then the tuple $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda_1^3 \to C$ which we depict imformally as a diagram



where the dotted arrows indicate the boundary of the "missing" face of the horn Λ^3_1. Using our assumption that C is an ∞-category, we can extend τ_0 to a 3-simplex τ of C. Then the face $d_1(\tau)$ is a homotopy from h to h''. ■

§3.6 The homotopy category of an ∞-category

Let C be an ∞-category. For every pair of objects $X, Y \in C$, we let $\mathrm{Hom}_{\mathrm{hC}}(X, Y)$ denote the set of homotopy classes of morphisms from X to Y in C. For every morphism $f : X \to Y$, we let $[f]$ denote its equivalence class in $\mathrm{Hom}_{\mathrm{hC}}(X, Y)$.

It follows from the 2 previous propositions that for every triple of objects $X, Y, Z \in C$, there is a unique composition law

$$\circ : \mathrm{Hom}_{\mathrm{hC}}(Y, Z) \times \mathrm{Hom}_{\mathrm{hC}}(X, Y) \to \mathrm{Hom}_{\mathrm{hC}}(X, Z)$$

satisfying the identity $[g] \circ [f] = [h]$ whenever $h : X \to Z$ is a composition of f and g in the ∞-category C.

Proposition 3.6.1

Let C be an ∞-category. Then

1. The composition law above is associative, that is for every triple of composable morphisms $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ in C, we have an equality $([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f])$ in $\mathrm{Hom}_{\mathrm{hC}}(W, Z)$.

2. For every object $X \in C$, the homotopy class $[\mathrm{id}_X] \in \mathrm{Hom}_{\mathrm{hC}}(X, X)$ is a two-sided identity with respect to the composition law above. That is, for every morphism $f : W \to X$ in C and every morphism $g : X \to Y$ in C, we have $[\mathrm{id}_X] \circ [f] = [f]$ and $[g] \circ [\mathrm{id}_X] = [g]$.

Definition 3.6.2 (The Homotopy Category)**.** Let C be an ∞-category. We define a category hC as follows:

- The objects of hC are the objects of C.
- For every pair of objects $X, Y \in C$, we let $\mathrm{Hom}_{\mathrm{hC}}(X, Y)$ denote the collection of homotopy classes of morphisms from X to Y in the ∞-category C (as discussed just above).
- For every object $X \in C$, the identity morphism from X to itself in hC is given by the homotopy class $[\mathrm{id}_X]$.
- Composition of morphisms is defined as above.

We will refer to hC as the homotopy category of the ∞-category C.

Example 3.6.3

Let C be an ordinary category. Then the homotopy category of the ∞-category $N_\bullet(C)$ can be identified with C. For instance, the homotopy category $h\Delta^n$ can be identified with $[n] = \{0 < 1 < \ldots < n\}$.

Example 3.6.4

Let X be a topological space, and consider the singular simplicial set $\mathrm{Sing}_\bullet(X)$ as an ∞-category. Then the homotopy category $\mathrm{hSing}_\bullet(X)$ can be identified with the fundamental groupoid $\pi_{\leq 1}(X)$.

In fact, we can regard the treatment of ∞-categories up to now when restricted to ∞-categories of the form $\mathrm{Sing}_\bullet(X)$ as providing a construction of the fundamental groupoid of X.

Definition 3.6.5. A morphism $f : X \to Y$ of an ∞-category C is an isomorphism if the homotopy class $[f]$ is an isomorphism in the homotopy category hC.

Proposition 3.6.6

Let C be a Kan Complex. Then every morphism in C is an isomorphism.

4 Back to Kan Complexes

§4.1 The Homotopy Category of Kan Complexes

Recall that we defined the category of simplicial sets as the functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$. This automatically yields morphisms between simplicial sets.

Let X, Y be simplicial sets and suppose we have $f, g : X \to Y$, which we identify with vertices of the simplicial set $\mathrm{Fun}(X, Y)$. We will say that f and g are homotopic if they belong to the same connected component of the simplicial set $\mathrm{Fun}(X, Y)$.

In other words,

Definition 4.1.1. Let X and Y be simplicial sets, and suppose we are given a pair of morphisms $f_0, f_1 : X \to Y$. A homotopy from f_0 to f_1 is a morphism $h : \Delta^1 \times X \to Y$ satisfying $f_0 = h|_{\{0\}\times X}$ and $f_1 = h|_{\{1\}\times X}$.

Definition 4.1.2 (The Homotopy Category of Kan Complexes)**.** We define a category hKan as follows:

- The objects of hKan are Kan complexes.
- If X and Y are Kan complexes, then $\mathrm{Hom}_{\mathrm{hKan}}(X, Y) = [X, Y] = \pi_0(\mathrm{Fun}(X, Y))$ is the set of homotopy classes of morphisms from X to Y.
- If X, Y, Z are Kan complexes, then the composition law

  $$\circ : \mathrm{Hom}_{\mathrm{hKan}}(Y, Z) \times \mathrm{Hom}_{\mathrm{hKan}}(X, Y) \to \mathrm{Hom}_{\mathrm{hKan}}(X, Z)$$

  is characterized by the formula $[g] \circ [f] = [g \circ f]$.

We will refer to hKan as the Homotopy category of Kan Complexes.

§4.2 Homotopy Equivalences and Weak Homotopy Equivalences

Definition 4.2.1. Let $f : X \to Y$ be a morphism of simplicial sets. We will say that a morphism $g : Y \to X$ is a homotopy inverse to f if the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity morphisms id_X and id_Y respectively.

We say that $f : X \to Y$ is a homotopy equivalence if it admits a homotopy inverse g.

Example 4.2.2

Let $f : X \to Y$ be a homotopy equivalence of topological spaces. Then the induced map of singular simplicial sets $\mathrm{Sing}_\bullet(f) : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(Y)$ is a homotopy equivalence.

§4.3 Homotopy Groups of a Kan Complex

Definition 4.3.1 (Pointed simplicial set)**.** A pointed simplicial set is a pair (X, x) where X is a simplicial set and x is a vertex of X. If X is a Kan complex, then we refer to the pair (X, x) as a pointed Kan complex.

A pointed map between two Kan Complexes $(X, x), (Y, y)$ is a morphism of Kan complexes $f : X \to Y$ satisfying $f(x) = y$. We let Kan_* denote the category whose objects are pointed Kan complexes and whose morphisms are pointed maps.

Definition 4.3.2 (Pointed homotopic)**.** Let (X, x) and (Y, y) be simplicial sets, and suppose we are given a pair of pointed maps $f, g : X \to Y$, which we identify with vertices of the simplicial set $\mathrm{Fun}(X, Y) \times_{\mathrm{Fun}(\{x\},Y)} \{y\}$.

We say that f and g are pointed homotopic if they belong to the same connected component of $\mathrm{Fun}(X, Y) \times_{\mathrm{Fun}(\{x\},Y)} \{y\}$.

Definition 4.3.3 (Homotopy Category of pointed Kan complexes)**.** We define a category hKan_* as follows:

- The objects of hKan_* are pointed Kan complexes (X, x).
- If (X, x) and (Y, y) are pointed Kan complexes, then $\mathrm{Hom}_{\mathrm{hKan}_*}((X, x),(Y, y)) = [X, Y]_*$ is the set of pointed homotopy classes of morphisms from (X, x) to (Y, y).
- If $(X, x),(Y, y),(Z, z)$ are pointed Kan complexes, then the composition law

  $$\circ : \mathrm{Hom}_{\mathrm{hKan}_*}((Y, y),(Z, z)) \times \mathrm{Hom}_{\mathrm{hKan}_*}((X, x),(Y, y)) \to \mathrm{Hom}_{\mathrm{hKan}_*}((X, x),(Z, z))$$

  is characterized by the formula $[g] \circ [f]g = [g \circ f]$.

We refer to hKan_* as the homotopy category of pointed Kan complexes.

Remark 4.3.4 — We are now ready to construct the homotopy groups of Kan complexes

Let X be a topological space and let $x \in X$ be a point. For every positive integer n, we let $\pi_n(X, x)$ denote the set of homotopy classes of pointed maps $(S^n, x_0) \to (X, x)$ where S^n denotes a sphere of dimension n and $x_0 \in S^n$ is a chosen base point. The set $\pi_n(X, x)$ can be endowed with the structure of a group, which we refer to as the nth homotopy group of X with respect to the base point x. Note that the sphere S^n can be realized as the quotient $|\Delta^n|/|\partial\Delta^n|$, obtained from the topological simplex $|\Delta^n|$ by collapsing its boundary to a single point.

We can therefore identify pointed maps $(S^n, x_0) \to (X, x)$ with maps of simplicial sets $f : \Delta^n \to \mathrm{Sing}_\bullet(X)$ which carry the boundary $\partial\Delta^n$ to the simplicial subset $\{x\} \subseteq \mathrm{Sing}_\bullet(X)$. This gives us a direct construction of the homotopy group $\pi_n(X, x)$ in terms of the simplicial set $\mathrm{Sing}_\bullet(X)$.

This construction can be applied directly to any Kan complex.

§4.4 Whitehead's Theorem

Theorem 4.4.1 (Whitehead's Theorem for Kan complexes)

Let $f : X \to Y$ be a morphism of Kan complexes. Then f is a homotopy equivalence if and only if it satisfies the following two conditions:

- The map of sets $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.
- For every vertex $x \in X$ having image $y = f(x)$ in Y and every positive integer n, the map of homotopy groups $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, y)$ is bijective.

Theorem 4.4.2

The geometric realization functor $| \bullet | : \mathrm{Set}_\Delta \to \mathrm{Top}$ induces an equivalence from the homotopy category hKan to the full subcategory of hTop spanned by those topological spaces X which have the homotopy type of a CW complex.

Proof. The proof of this major theorem consists of three major steps:

1. Show that geometric realization is well-defined at the level of homotopy categories
2. Show that the geometric realization functor $| \bullet | : \mathrm{hKan} \to \mathrm{hTop}$ is fully faithful.
3. Show that if Y is a topological space, then the counit map $v_Y : |\mathrm{Sing}_\bullet(Y)| \to Y$ is a homotopy equivalence if and only if Y has the homotopy type of a CW complex.

■