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MODEL CATEGORIES THAT ARE NOT QUILLEN EQUIVALENT

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We introduce the notion of the model category and the homotopy category associated to a model category, followed by relevant examples of model categories. We introduce the notion of Quillen equivalence, discuss its importance and draw a contrast between Quillen equivalence and categorical equivalence. In particular, we highlight how a Quillen equivalence behaves under association of $(\infty, 1)$-categories to model categories. We introduce two model categories that have equivalent homotopy categories but fail to be Quillen equivalent, based on a paper by Dugger & Shipley. We show that these two model categories have equivalent homotopy categories, filling in some missing details in the paper by Dugger & Shipley, and give an overview of how to show that the two model categories are not Quillen equivalent.

§1 Introduction

We assume that the reader has a rudimentary working knowledge of category theory and is familiar with the basic definitions and concepts in category theory. The first section attempts to introduce as much as is reasonable of the machinery from category theory related to liftings that shows up in later sections of this work.

Following a short exposition of the basic notions from category theory, model categories are introduced along with the notion of associating a homotopy category to a model category. We provide multiple examples, some of them serving as "toy" examples while others are more important, such as the model structure for the category of topological spaces and a model structure for modules. The latter will be the main model structure that we will work with in subsequent sections of this work. We also highlight how the notion of a model structure for a given category can be weakened, in particular we define homotopical categories and categories with weak equivalences and discuss how these weaker assumptions manifest themselves for the homotopy categories associated to homotopical categories and categories with weak equivalences.

Subsequently, we define Quillen functors, Quillen adjunctions and Quillen equivalences. In our definitions we provide multiple equivalent conditions and we show that these conditions are indeed equivalent. We construct an example of a map between topological spaces that is a weak equivalence for one model structure on the category of topological spaces that is not a weak equivalence for another model structure on the category of topological spaces. We discuss briefly how Quillen equivalences can be turned into an equivalence of $(\infty, 1)$-categories, by touching on how to associate an $(\infty, 1)$-category to a model category. This process involves the hammock localization, a cofibrantly generated model structure for the category of simplicial categories, fibrant replacement and taking the homotopy coherent nerve of a simplicial subcategory.

In the last section of this paper, we concern ourselves with the study of two particular model categories that have equivalent homotopy categories but fail to be Quillen equivalent. The first proof of non Quillen equivalence of these two model categories involved K-theory and was computationally involved. In their paper [7], Dugger & Shipley presented a different way of proving the non Quillen equivalence of these two model categories. Our aim is to provide a full and rigorous proof that the homotopy categories of the two chosen model categories are categorically equivalent and subsequently provide as detailed of an exposition as is reasonable within the confines of a master thesis of the argument for the non Quillen equivalence used by Dugger & Shipley.

§2 Basic category theory

As mentioned in the introduction, we will try to present as much of the required category theory for liftings and their properties as is feasible and sensible. Of particular interest is the small object argument, which we introduce at the end of this section.

If the reader wishes to consult a general reference on all matters related to category theory, we can wholeheartedly recommend [14]. We would be remiss if we did not advise caution in how one approaches *Categories and Sheaves*. Due to the detailed nature of this book, we caution the reader that it can be overwhelming when taking their first steps in category theory.

Throughout the whole thesis, we chose keep all definitions concerning general category theoretical concepts in line with nLab [16], which has become somewhat of a reference for all matters related to category theory.

Definition 2.1. A category $\mathcal{C}$ is a *small category* if its objects form a set and all hom sets $\hom(A, B)$ for objects $A, B \in \mathcal{C}$ are sets.

Definition 2.2. A category $\mathcal{C}$ is a *locally small category* if all hom sets $\hom(A, B)$ for objects $A, B \in \mathcal{C}$ are sets.

Remark. We will distinguish generic hom sets that may potentially be proper classes instead of sets from hom sets that are sets by denoting the former by $\hom(A, B)$ and the latter $\mathrm{Hom}(A, B)$.

Definition 2.3. Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a category. The *functor category* $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ is a category whose objects are given by functors $F : C \to D$ and morphisms are given by natural transformations between such functors.

Remark. Some authors refer to the functor category $\mathrm{Func}(\mathcal{C}, \mathcal{D})$ as the *category of diagrams in $\mathcal{D}$ with space of* $\mathcal{C}$ and denote it by $[\mathcal{C}, \mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$.

Definition 2.4. For any category $\mathcal{C}$ we define its *arrow category* $\mathrm{Arr}(\mathcal{C})$ as the category whose objects are morphisms $a : X \to Y$ of $\mathcal{C}$ and whose morphisms are given by commutative squares, i.e. a morphism $f : a \to b$ in $\mathrm{Arr}(\mathcal{C})$ is given by the following commutative square, where $a : X \to Y$, $b : \tilde{X} \to \tilde{Y}$, $f_0 : X \to \tilde{X}$ and $f_1 : Y \to \tilde{Y}$ are morphisms of $\mathcal{C}$,

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[d, "a"'] & \tilde{X} \arrow[d, "b"] \\ Y \arrow[r, "f_1"'] & \tilde{Y} \end{tikzcd}$$

and the composition of two morphisms $f : a \to b$, $g : b \to c$ is given by putting commutative squares side by side to get a new commutative square $gf : a \to c$ as follows.

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[d, "a"'] & \tilde{X} \arrow[r, "g_0"] \arrow[d, "b"] & \bar{X} \arrow[d, "c"] \\ Y \arrow[r, "f_1"'] & \tilde{Y} \arrow[r, "g_1"'] & \bar{Y} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g_0 f_0"] \arrow[d, "a"'] & \bar{X} \arrow[d, "c"] \\ Y \arrow[r, "g_1 f_1"'] & \bar{Y} \end{tikzcd}$$

Remark. We can also define the arrow category of a category C as the functor category $\mathrm{Func}(\mathcal{D}, C)$, where $\mathcal{D} = \{0 \to 1\}$ is the interval category, that is to say a category with two objects and a single non-identity morphism between them.

Definition 2.5. A functor $F : C \to \mathcal{D}$ is said to be *full* if for any two objects X, Y of C the map $\hom_C(X, Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$ given by F is an epimorphism.

Remark. It is important to note that a full functor between two categories $F : C \to \mathcal{D}$ does not have to be an epimorphism in the category of categories, since we do not require $F : \mathrm{obj}(C) \to \mathrm{obj}(\mathcal{D})$ to be surjective. We will now show that it would have to be so.

We consider the category of small categories and two small categories C and $\mathcal{D}$. Suppose $F : C \to \mathcal{D}$ is an epimorphism in the category of small categories that is not surjective on objects. Let $\tilde{\mathcal{D}} \to \mathcal{D}$ be the full subcategory of $\mathcal{D}$ whose objects are given by $F(\mathrm{obj}(C))$ and all morphisms of $\mathcal{D}$ with source and target object that is in $F(\mathrm{obj}(C))$. Since the category of small categories has all small limits and colimits we can consider the pushout (the definition of the pushout can be found in 2.29)

$$\begin{tikzcd}
\tilde{\mathcal{D}} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathcal{D} \arrow[r] & \mathcal{D} \sqcup_{\tilde{\mathcal{D}}} \mathcal{D}
\end{tikzcd}$$

We have two functors $G, H : \mathcal{D} \to \mathcal{D} \sqcup_{\tilde{\mathcal{D}}} \mathcal{D}$.The functor F factors through $\tilde{\mathcal{D}}$, which means that there exists a functor $K : C \to \tilde{\mathcal{D}}$ such that $F = iK$, where $i : \tilde{\mathcal{D}} \to \mathcal{D}$ is the inclusion of the full subcategory $\tilde{\mathcal{D}}$ into $\mathcal{D}$.

Clearly $GF = GiK$ and $HF = HiK$. By the universal property of the pushout, we have $GF = HF$, but no matter how the functors G and H map $\mathcal{D}$ to $\mathcal{D} \sqcup_{\tilde{\mathcal{D}}} \mathcal{D}$, we cannot have $H = G$, since there exists at least one object X of $\mathcal{D}$ such that $X \notin i(\tilde{\mathcal{D}})$, which implies that $G(X) \neq H(X)$. Therefore, $GF = HF$ does not imply that $G = H$ and therefore F cannot be an epimorphism.

Definition 2.6. A functor $F : C \to \mathcal{D}$ is said to be *faithful* if for any two objects X, Y of C the map $\hom_C(X, Y) \to \hom_{\mathcal{D}}(F(X), F(Y))$ given by F is a monomorphism.

Definition 2.7. Let $\mathcal{D}$ be a subcategory of C. We say that $\mathcal{D}$ is a full subcategory of C if for any two objects X and Y in $\mathcal{D}$ we have $\hom_C(X, Y) = \hom_{\mathcal{D}}(X, Y)$.

Equivalently, $\mathcal{D}$ is a full subcategory if the inclusion functor $i : \mathcal{D} \to C$ is full.

Definition 2.8. A *retract* of an object Y of a category C is an object X for which there exist morphisms $i : X \to Y$ and $r : Y \to X$ such that $ri = 1_X$.

Definition 2.9. A *retract* of a morphism $f : A \to B$ of a category C is a morphism $g : X \to Y$ such that f is a retract of g as objects in the category $\mathrm{Arr}(C)$.

The following proposition is a very basic but extremely important and heavily used result. While its proof is comparatively trivial, we decided to show the result in full detail for the sake of completeness.

Proposition 2.10. A morphism $f : A \to B$ of a category $\mathcal{C}$ is a retract of a morphism $g : X \to Y$ if and only if there exists a commutative diagram of the following form

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & X \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r, "j"'] \arrow[rr, bend right, "1_B"'] & Y \arrow[r, "p"'] & B
\end{tikzcd}$$

Proof. Suppose $f : A \to B$ is a retract of $g : X \to Y$ as objects of $\mathrm{Arr}(\mathcal{C})$. This means that there exist morphisms $i : f \to g$ and $r : g \to f$ in the form of the following two commutative squares

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r, "i_1"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "r_0"] \arrow[d, "g"'] & A \arrow[d, "f"] \\
Y \arrow[r, "r_1"'] & B
\end{tikzcd}$$

such that $ri = 1_f$. This means that $r_0 i_0 = 1_A$ and $r_1 i_1 = 1_B$ and thus we get the commutative diagram in the statement of the proposition as desired.

Conversely, suppose we are given the commutative diagram from the statement of the proposition. We want to show that f is a retract of g when considering f and g as objects in $\mathrm{Arr}(\mathcal{C})$. By splitting the diagram into two commutative squares, we end up with two morphisms $i : f \to g$ and $r : g \to f$ of $\mathrm{Arr}(\mathcal{C})$, each represented as a commutative square in the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "f"'] \arrow[rrr, bend left, "1_A"] & X \arrow[r, dotted, leftrightarrow, "1_X"] \arrow[d, "g"] & X \arrow[r, "r_0"] \arrow[d, "g"'] & A \arrow[d, "f"] \\
B \arrow[r, "i_1"'] \arrow[rrr, bend right, "1_B"'] & Y \arrow[r, dotted, leftrightarrow, "1_Y"] & Y \arrow[r, "r_1"'] & B
\end{tikzcd}$$

It is obvious that we can extract a commutative square $ri : f \to f$ from above diagram:

$$\begin{tikzcd}
A \arrow[r, "r_0 i_0"] \arrow[d, "f"'] & A \arrow[d, "f"] \\
B \arrow[r, "r_1 i_1"'] & B
\end{tikzcd}$$

It is obvious from the previous diagram that $r_0 i_0 = 1_A$ and $r_1 i_1 = 1_B$ and therefore $ri = 1_f$, which shows that starting from the commutative diagram in the statement of the proposition we can construct appropriate morphisms $i : f \to g$ and $r : g \to f$ such that $ri = 1_f$, which means that f is indeed a retract of g when considered as objects of $\mathrm{Arr}(\mathcal{C})$. ∎

Definition 2.11. Let $F : \mathcal{A} \to C$ and $G : \mathcal{B} \to C$ be functors. The *comma category* of F over G, denoted by (F/G) is a category whose objects are triples (A, B, f) with $A \in \mathcal{A}$, $B \in \mathcal{B}$, $f \in \hom_C(F(A), F(B))$ and morphisms between triples (A, B, f), (A', B', f') are given by pairs (α, β) where $\alpha \in \hom_{\mathcal{A}}(A, A')$ and $\beta \in \hom_{\mathcal{B}}(B, B')$ and the following diagram commutes.

$$\begin{tikzcd} F(A) \arrow[r, "F(\alpha)"] \arrow[d, "f"'] & F(A') \arrow[d, "f'"] \\ G(B) \arrow[r, "G(\beta)"'] & G(B') \end{tikzcd}$$

Definition 2.12. Let C be a category and a an object of C. The *slice category* of C over a, denoted by (C/a), is the comma category $(1_C/F)$, where 1_C is the identity functor and $F : 1 \to C$ with $F(1) = a$.

Definition 2.13. Let C be a category and a an object of C. The *coslice category* of C under a, denoted by (a/C), is the comma category $(F/1_C)$, where 1_C is the identity functor and $F : 1 \to C$ with $F(1) = a$.

Definition 2.14. Let C be a category and $\mathcal{D}$ a small category. We define the *diagonal functor* $\Delta : C \to C^{\mathcal{D}}$ which takes all objects of $X \in C$ to $\Delta_X : \mathcal{D} \to C$, which is the constant functor that sends all objects of $\mathcal{D}$ to X and all morphisms of $\mathcal{D}$ to 1_X.

Δ sends each morphism $X \xrightarrow{f} Y$ of C to the natural transformation $\eta(f) : \Delta_X \to \Delta_Y$ given by $\eta(f)(Z) = f$ for all $Z \in \mathcal{D}$.

Definition 2.15. Let $F : C \to \mathcal{D}$ be a functor. A *cone* over F is an object X along with a natural transformation $\eta : \Delta_X \to F$.

Definition 2.16. Let $F : C \to \mathcal{D}$ be a functor. A *cocone* under F is an object X along with a natural transformation $\eta : F \to \Delta_X$.

Definition 2.17. A category C is called a *filtered category* if

1. C is non-empty, that is to say it contains at least one object,
2. for any two object X, Y of C, there exists an object Z and morphisms $X \to Z$ and $Y \to Z$,
3. for any two parallel morphisms $f, g : X \to Y$ between two objects X and Y of C, there exists a third object Z along with a morphism $h : Y \to Z$ such that $hf = hg$.

Definition 2.18. Let C be a category and $f : A \to B$ and $g : X \to Y$ be morphisms. We say that f has the *left lifting property* with respect to g and g has the *right lifting property* with respect to f if for every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

there exists a lift $h : B \to X$, such that the following diagram is commutative.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, "h"] & Y \end{tikzcd}$$

Lemma 2.19. Let $\mathcal{C}$ be a category and $f = pi$ a factorization in $\mathcal{C}$. If f has the left lifting property with respect to p, then f is a retract of i and dually if f has the right lifting property with respect to i, then f is a retract of p.

Proof. Suppose f has the left lifting property with respect to p. This means that we have a lift $r : B \to C$, as can be seen in the following diagram.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[d, "p"] \\ B \arrow[ur, "r"] \arrow[r, leftrightarrow, dashed, "1_B"] & B \end{tikzcd}$$

By commutativity of this diagram we can produce the following commutative diagram, which exhibits f as a retract of i.

$$\begin{tikzcd} A \arrow[r, leftrightarrow, dotted, "1_A"] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & A \arrow[r, leftrightarrow, dotted, "1_A"] \arrow[d, "i"] & A \arrow[d, "f"] \\ B \arrow[r, "r"] \arrow[rr, bend right, "1_B"'] & C \arrow[r, "p"] & B \end{tikzcd}$$

where $pr = 1_B$ follows from $prf = f$ as evident from the diagram.

The second part of the lemma can be shown using a similar reasoning. ■

Definition 2.20. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between a small category $\mathcal{C}$ and a category $\mathcal{D}$. A *colimit* for F is an object $X \in \mathcal{D}$ along with a natural transformation $\eta : F \to \Delta X$ such that for all objects $Y \in \mathcal{D}$ and natural transformation $\tilde{\eta} : F \to \Delta Y$ there exists a unique morphism $\varphi : X \to Y$ in $\mathcal{D}$ for which $\Delta(\varphi)\eta = \tilde{\eta}$.

If $\mathcal{C}$ is a filtered category, then we speak of a filtered colimit.

Remark. Any two colimits for a functor F are canonically isomorphic, which is why we refer any colimit of a functor as *the colimit* as long as we know that it exists.

Usually the colimit of a functor F is denoted by $\varinjlim F$.

Definition 2.21. The *limit* of a functor $F : \mathcal{C} \to \mathcal{D}$ is the colimit of the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

More explicitly, this means that a limit for a functor $F : \mathcal{C} \to \mathcal{D}$ is given by an object $X \in \mathcal{D}$ and a natural transformation $\eta : \Delta X \to F$ such that for all $Y \in \mathcal{D}$ and natural transformations $\tilde{\eta} : \Delta Y \to F$ there exists a unique morphism $\varphi : Y \to X$ in $\mathcal{D}$ such that $\eta\Delta(\varphi) = \tilde{\eta}$.

Since a major component of this thesis is the notion of Quillen equivalence, we deem it sensible to recall the definition of adjoint functors at this point.

Definition 2.22. Let $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to C$ be functors. An adjunction from F to G is defined as a collection of isomorphisms

$$\eta_{X,Y} : \hom_{\mathcal{D}}(F(X), Y) \to \hom_C(X, G(Y)),$$

for any $X \in C$, $Y \in \mathcal{D}$, that are natural in X and Y.

If an adjunction from F to G exists, we say that F and G are *adjoint functors.* F is the left adjoint of G and G is the right adjoint of F.

Definition 2.23. Let $(L, R) : C \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ be an adjunction. We call the natural transformation $\eta : \mathrm{id}_C \to R \circ L$ the *unit of the adjunction* (L, R) and the natural transformation $\epsilon : L \circ R \to \mathrm{id}_{\mathcal{D}}$ the *counit of the adjunction.*

Definition 2.24. A category C is said to have all *small limits (colimits)* if the limit (colimit) for any functor F from a small category $\mathcal{D}$ to C exists.

Such a category C is called a *complete* category if all small limits exist, a *cocomplete* category if all small colimits exist and a *bicomplete* category if it is both complete and cocomplete.

If a category C has all finite limits (colimits), that is to say that the limit (colimit) for any functor F from a finite category $\mathcal{D}$ to C exists, then we say that C is a *finitely complete* (*finitely cocomplete*) category.

Remark. Given a category C with all colimits and a class of morphisms S of C, we denote by

- $\mathrm{rlp}(S)$ morphisms with the right lifting property with respect to morphisms in S,
- $\mathrm{llp}(S)$ morphisms with the left lifting property with respect to morphisms in S.

Proposition 2.25. Let C be any category and S a class of maps of C. Then $\mathrm{llp}(S)$ and $\mathrm{rlp}(S)$ are closed under retracts and composition.

Proof. Suppose $f : A \to B$ and $g : B \to C$ are maps with the left lifting property with respect to maps in a class of maps S. Let $\psi := gf : A \to C$.

We need to show that for any given diagram of the form

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\psi"'] & X \arrow[d, "\varphi \in S"] \\ C \arrow[r, "\beta"'] & Y \end{tikzcd}$$

there exists a lift $h : C \to X$.

We split this diagram into a bigger commutative diagram.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & X \arrow[dd, "\varphi \in S"] \\ B \arrow[d, "g"'] \arrow[dr, "\beta \circ g"'] & \\ C \arrow[r, "\beta"'] & Y \end{tikzcd}$$

Since $f \in llp(S)$ we know that there exists a lift $h : B \to X$ and we get a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & X \arrow[d, "\varphi \in S"] \\ B \arrow[r, "\beta \circ g"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

Since $g \in llp(S)$, there exists a lift $j : C \to X$ such that the following diagram is commutative.

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[d, "\varphi \in S"] \\ C \arrow[r, "\beta"'] \arrow[ur, "j"] & Y \end{tikzcd}$$

We need to verify that j is such that the following diagram is commutative.

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[d, "\varphi \in S"] \\ C \arrow[r, "\beta"'] \arrow[ur, "j"] & Y \end{tikzcd}$$

The only thing we need to verify is that $\alpha = j \circ g \circ f$. By commutativity of the diagrams above we know that $\alpha = h \circ f$ and $h = j \circ g$. Thus we have $\alpha = h \circ f = j \circ g \circ f$ and we are done.

The same argument can be used to show that the class $rlp(S)$ is closed under composition.

Suppose that we have the following commutative square, where f is a retract of g as before.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & E \arrow[d, "\varphi"] \\ B \arrow[r] & F \end{tikzcd}$$

Since f is a retract of g, we have the following commutative diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_A"] & X \arrow[r] \arrow[d, "g"] & A \arrow[r] \arrow[d, "f"] & E \arrow[d, "\varphi"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & Y \arrow[r] & B \arrow[r] & F \end{tikzcd}$$

It is clear from this diagram that any commutative square involving $f : A \to B$ and $\varphi : E \to F$ with $\varphi \in S$ can be turned into a commutative square involving $g : X \to Y$ and $\varphi : E \to F$. Thus any commutative diagram involving f and φ is such that we can embed a commutative square involving g in it and use the left lifting property of g to produce a lift for the commutative square involving f, which exhibits the left lifting property of f with respect to φ.

Closure under retracts of $rlp(S)$ is formally dual to the above argument. ■

Definition 2.26. Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a category whose objects are given by a set $\mathcal{I}$ and whose only morphisms are identity morphisms.

Any functor $F : \mathcal{D} \to \mathcal{C}$ can be characterized by a collection $\{C_i\}_{\mathcal{I}}$, where each $C_i \in \mathcal{C}$.

The *coproduct* of $\{C_i\}_{\mathcal{I}}$ is given by the colimit of F and denoted by $\coprod\limits_i C_i$.

Definition 2.27. Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a category whose objects are given by a set $\mathcal{I}$ and whose only morphisms are identity morphisms.

Any functor $F : \mathcal{D} \to \mathcal{C}$ can be characterized by a collection $\{C_i\}_{\mathcal{I}}$, where each $C_i \in \mathcal{C}$.

The *product* of $\{C_i\}_{\mathcal{I}}$ is given by the limit of F and denoted by $\prod\limits_i C_i$.

Definition 2.28. Let $\mathcal{C}$ be a category. An object $\emptyset$ of $\mathcal{C}$ is called an *initial object* if there exists exactly one morphism from $\emptyset$ to each object of $\mathcal{C}$.

An object $*$ of $\mathcal{C}$ is called a *terminal object* if there exists exactly one morphism from each object of $\mathcal{C}$ to $*$.

Definition 2.29. Let $\mathcal{C}$ be a category and suppose we have the following diagram

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B & & C \end{tikzcd}$$

where A, B, C are objects of $\mathcal{C}$.

The colimit of this diagram is called the *pushout*.

Remark. If in above definition the colimit exists, then the pushout of the diagram is the following commutative square.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B \arrow[dr, "i_1"'] & & C \arrow[dl, "i_2"] \\ & X & \end{tikzcd}$$

Additionally, the pushout of a diagram has the universal property that for any commutative square of the form

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B \arrow[dr, "j_1"'] & & C \arrow[dl, "j_2"] \\ & Y & \end{tikzcd}$$

there exists a unique morphism $h : X \to Y$ such that $hi_1 = j_1$ and $hi_2 = i_2$, which is summarized by the following commutative diagram.

$$\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
B \arrow[dr, "i_1"] \arrow[ddr, "j_1"'] & & C \arrow[dl, "i_2"'] \arrow[ddl, "j_2"] \\
 & X \arrow[d, "h"] & \\
 & Y &
\end{tikzcd}$$

Definition 2.30. Let C be a category and suppose we have the following diagram

$$\begin{tikzcd}
B \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
 & A &
\end{tikzcd}$$

where A, B, C are objects of C.

The limit of this diagram is called the *pullback*.

Remark. If the limit of a diagram as in above definition exists, then the pullback is a commutative square of the following form.

$$\begin{tikzcd}
 & X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
B \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
 & A &
\end{tikzcd}$$

Not unlike for pushouts, pullbacks have the universal property that for any commutative square

$$\begin{tikzcd}
 & Y \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
B \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
 & A &
\end{tikzcd}$$

there exists a unique morphism $h : Y \to X$ such that $p_1 h = r_1$ and $p_2 h = r_2$, which we summarize with the following diagram.

$$\begin{tikzcd}
 & Y \arrow[ddl, "r_1"'] \arrow[d, "h"] \arrow[ddr, "r_2"] & \\
 & X \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
B \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\
 & A &
\end{tikzcd}$$

Definition 2.31. Let C be a category with all small colimits and α an ordinal. An *α-sequence* is a colimit-preserving functor $X : \alpha \to C$, which we can present as

$$X_0 \to X_1 \to \ldots \to X_\beta \to \ldots$$

where $\beta < \alpha$ and $X_\gamma := X(\gamma)$ for any $\gamma < \alpha$.

Since X preserves colimits, the induced map $\varinjlim_{\beta<\gamma} X_\beta \to X_\gamma$ is an isomorphism for any $\gamma < \alpha$.

The map $X_0 \to \varinjlim_{\beta<\alpha} X_\beta$ is called the *composition of the α-sequence*, which is unique up to isomorphism of X.

If I is a collection of morphisms of C such that for any $\beta + 1 < \alpha$, the map $X_\beta \to X_{\beta+1}$ is in I, then the composition of the α-sequence X is a *transfinite composition of maps* or to be more precise a *transfinite composition of maps of I*.

Definition 2.32. Let γ be a cardinal and α an ordinal. We say that α is *γ-filtered* if α is a limit ordinal and if for any $A \subseteq \alpha$ with $|A| \leq \gamma$, then $\sup A < \alpha$.

Definition 2.33. Let C be a category with all small colimits, I a collection of morphisms of C, Y an object of C and κ a cardinal. The object Y is said to be *κ-small relative to I* if for all κ-filtered ordinals α and all α-sequences $X : \alpha \to C$

$$X_0 \to X_1 \to \ldots \to X_\beta \to \ldots$$

the maps $X_\beta \to X_{\beta+1}$ is in the collection I for any $\beta + 1 < \alpha$ and the induced map of morphisms

$$\varinjlim_{\beta<\alpha} \hom_C(Y, X_\beta) \to \hom_C(Y, \varinjlim_{\beta<\alpha} X_\beta)$$

is an isomorphism.

If Y is κ-small relative to I for some κ, then Y is said to be *small relative to I* and we say that Y is *small* if Y is small relative to the collection of morphisms of C.

Remark. A more intuitive way of thinking of transfinite composition is summarized in the following diagram.

$$\begin{tikzcd}
X(0) \arrow[r, "X(0\to1)"] \arrow[dr] & X(1) \arrow[r, "X(1\to2)"] \arrow[d] & X(2) \arrow[r] \arrow[dl] & \ldots \arrow[dll] \\
 & X(\alpha) & &
\end{tikzcd}$$

Definition 2.34. Let C be a category that has all colimits and J a subset of morphisms of C. A *J-cell complex* in C is an object that is connected to the initial object by a transfinite composition of pushouts of morphisms in J. By this we mean that $f : X \to Y$ is a J-cell complex in C if there exists an ordinal α and an α-indexed transfinite sequence $X : \alpha \to C$ such that f is the transfinite composition of the transfinite sequence X and such that for any β with $\beta+1 < \alpha$, the map $X(\beta \to \beta+1)$ can be obtained as the pushout of a morphism $J \ni g_\beta : C_\beta \to D_\beta$, that is to say that there exists a pushout square

$$\begin{tikzcd} C_\beta \arrow[r] \arrow[d, "g_\beta"'] & X(\beta) \arrow[d, "X(\beta\to\beta+1)"] \\ D_\beta \arrow[r] & X(\beta+1) \end{tikzcd}$$

A *relative J-cell complex* in C relative to some object A is defined as above with the exception that we are interested in the object being connected to A instead of the initial object.

Definition 2.35. Let C be a category with all colimits and let S be a class of morphisms of C. We denote by

- $\mathrm{cell}(S)$ the relative cell complexes obtained by transfinite composition of pushouts of coproducts of elements in S,
- $\mathrm{cof}(S)$ the class of retracts of elements of $\mathrm{cell}(S)$,
- $\mathrm{inj}(S)$ the class of morphisms with the right lifting property with respect to all morphisms in S and we call elements of $\mathrm{inj}(S)$ S-injective morphisms.

Definition 2.36. A category C is called a *pointed category* if it has a zero object, that is the category has an initial object and a terminal object which are isomorphic.

Theorem 2.37 (Small object argument)**.** Let C be a category and I a set of morphisms of C such that C has all colimits and each morphism in I has a small domain relative to transfinite composition of pushouts of morphisms in I, that is to say that the domains of the maps of I are small relative to the collection of maps $\mathrm{cell}(I)$.

Then any morphism f of C can be expressed as a factorization $f = hg$, where $h \in \mathrm{rlp}(I)$ and $g \in \mathrm{cell}(I)$, where $\mathrm{cell}(I)$ is the set of transfinite compositions of pushouts of morphisms in I and $\mathrm{rlp}(I)$ is the set of morphisms that have the right lifting property with respect to I.

Remark. The small object argument is a fundamental result in category theory. The name comes from Quillen who as far as we are aware was the first to use a simplified version of the more general small object argument we gave in Theorem 2.37 above, in [17] Chapter II.3, Lemma 3.

For a more recent paper that discusses the small object argument, we recommend [11] by Richard Garner, which may be of interest to the reader. In particular, this paper discusses the lack of universal property of the small object argument, issues concerning convergence and not having an obvious direct relation with other transfinite constructions in categorical algebra and provides a different treatment that addresses the aforementioned deficiencies.

For a proof of the small object argument in the formulation given above, we refer to [13], Theorem 2.1.14.

§3 Model category

§3.1 The definition of a model category

Definition 3.1. A *model category* is a category $\mathcal{M}$ along with three distinguished classes of maps $\mathcal{W}$, $\mathcal{F}$, $\mathcal{C}$, called weak equivalences, fibrations and cofibrations, respectively, such that the following axioms are satisfied.

1. $\mathcal{M}$ has all small limits and colimits,
2. for any two morphisms f, g such that the composition gf is defined, it holds that if two of f, g, gf are weak equivalences, then so is the third, i.e. if two of f, g, gf are in $\mathcal{W}$, then so is the third,
3. given a retract f of a fibration, cofibration or weak equivalence g, then f is also a fibration, cofibration or a weak equivalence,
4. cofibrations have the left lifting property with respect to morphisms in $\mathcal{W} \cap \mathcal{F}$ and fibrations have the right lifting property with respect to morphisms in $\mathcal{W} \cap \mathcal{C}$,
5. there exist two functorial factorizations (a, b) and (c, d) on $\mathcal{M}$ such that for any morphism $f : X \to Y$ in $\mathcal{M}$ we have $af \in \mathcal{C}$, $bf \in \mathcal{W} \cap \mathcal{F}$, $cf \in \mathcal{W} \cap \mathcal{C}$, and $df \in \mathcal{F}$ and such that given any commutative square

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"'] & \tilde{X} \arrow[d, "g"] \\ Y \arrow[r, "\psi"'] & \tilde{Y} \end{tikzcd}$$

there exist morphisms $(a, b)(\varphi, \psi)$, $(c, d)(\varphi, \psi)$ such that the following two diagrams commute

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "af"'] & \tilde{X} \arrow[d, "ag"] \\ Z \arrow[r, "(a,b)(\varphi,\psi)"'] \arrow[d, "bf"'] & \tilde{Z} \arrow[d, "bg"] \\ Y \arrow[r, "\psi"'] & \tilde{Y} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "cf"'] & \tilde{X} \arrow[d, "cg"] \\ Z \arrow[r, "(c,d)(\varphi,\psi)"'] \arrow[d, "df"'] & \tilde{Z} \arrow[d, "dg"] \\ Y \arrow[r, "\psi"'] & \tilde{Y} \end{tikzcd}$$

and such that $(a, b)(\varphi \circ \tilde{\varphi}, \psi \circ \tilde{\psi}) = (a, b)(\varphi, \psi) \circ (a, b)(\tilde{\varphi}, \tilde{\psi})$ and $(c, d)(\varphi \circ \tilde{\varphi}, \psi \circ \tilde{\psi}) = (c, d)(\varphi, \psi) \circ (c, d)(\tilde{\varphi}, \tilde{\psi})$.

Remark. Some authors introduce the notion of model categories by first introducing the notion of model structures. One motivation for doing so comes from the fact that it is possible to impose two different model structures on the same category. For completeness we also present this approach to defining the notion of model categories. It is trivial to see that these two ways of defining model categories are compatible.

Definition 3.2. A *model structure* on a bicomplete category $\mathcal{M}$ consists of a triple of distinguished classes of morphisms $(\mathcal{W}, \mathcal{F}, \mathcal{C})$, called weak equivalences, fibrations, and cofibrations respectively, such that conditions 2, 3, 4 and 5 in Definition 3.1 are satisfied.

Definition 3.3. A *model category* $\mathcal{M}$ is a bicomplete category along with a model structure $(\mathcal{W}, \mathcal{F}, \mathcal{C})$.

Remark. As we shall see later, there are other ways of defining model categories. In particular, we will come across one more equivalent way of defining model categories when we introduce the notion of categories with weak equivalences.

Remark. Model categories were first introduced by Quillen [17], who referred to them as closed model categories. It has become common to refer to them as simply model categories.

Definition 3.4. Let $\mathcal{M}$ be a model category with weak equivalences $\mathcal{W}$, fibrations $\mathcal{F}$ and cofibrations $\mathcal{C}$. We say that a morphism is

- an acyclic fibration or trivial fibration if it is in $\mathcal{W} \cap \mathcal{F}$,
- an acyclic cofibration or trivial cofibration if it is in $\mathcal{W} \cap \mathcal{C}$.

Remark. Since model categories are by definition bicomplete, any model category has an initial and a terminal object, given by the colimit and limit of the empty diagram.

We can also turn any model category $\mathcal{M}$ into a pointed model category $\mathcal{M}_*$, which is nothing more than a minimally modified version of $\mathcal{M}$ under the terminal object $*$. Objects of $\mathcal{M}_*$ are maps $* \to X$, where $X \in \mathcal{M}$ and morphisms between $* \to X$ and $* \to Y$ are morphisms $X \to Y$ that sends the unique map $* \to X$ to the unique map $* \to Y$.

Not every object X of $\mathcal{M}$ may come with a morphism $* \to X$ in the model category $\mathcal{M}$. This does not pose a big concern, seeing as model categories naturally come with a coproduct. Each object X of $\mathcal{M}$ has a coproduct with $*$, which allows us to consider the slightly modified model category $\tilde{\mathcal{M}}$ whose objects are given by $X \coprod *$, where X is an object of $\mathcal{M}$ and $*$ is the terminal object of $\mathcal{M}$. We have a functor that sends $X \coprod *$ of $\tilde{\mathcal{M}}$ to X of $\mathcal{M}$.

Definition 3.5. Let $\mathcal{M}$ be a category with a terminal object $*$. Its corresponding *category of pointed objects* is the category whose objects are given by morphisms of the form $* \to X$ and morphisms are given by commuting triangles of the form

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

Equivalently, we can define the category of pointed objects as the category $\mathcal{M}$ under the terminal object $*$ or the coslice category $(*/\mathcal{M})$.

Remark. Let $\mathcal{M}$ be a category with a terminal object $*$ and finite colimits. Then there is a forgetful functor $U : (*/\mathcal{M}) \to \mathcal{M}$ which has a left adjoint given by the object-wise coproduct with the base point $*$

$$(-) \coprod * : \mathcal{M} \to (*/\mathcal{M}).$$

Proposition 3.6. Let $\mathcal{M}$ be a category with a terminal object $*$ and finite colimits. Then $(*/\mathcal{M})$ has a zero object.

Proposition 3.7. Let $\mathcal{M}$ be a model category. Then there exists a model structure for the category $\mathcal{M}_*$ in which a map f is a cofibrant, fibration or weak equivalence if and only if Uf is a cofibration, fibration or weak equivalence respectively.

Proof. Weak equivalences in $\mathcal{M}_*$ clearly satisfy the 2-out-of-3 property. Closure under retracts for cofibrations, fibrations and weak equivalences is also trivially true. That means we only have to show that cofibrations have the left lifting property with respect to trivial fibrations, fibrations have the right lifting property with respect to trivial cofibrations and that there exist two functorial factorizations that satisfy the conditions given in the definition of a model category.

Let i be a cofibration and p a trivial fibration in $\mathcal{M}_*$. This means that by definition Ui is a cofibration and Up is a trivial fibration in $\mathcal{M}$. Therefore Ui has the left lifting property with respect to Up. Therefore there exists a lift. That lift has to preserve the basepoint, since all morphisms of $\mathcal{M}_*$ preserve the basepoint $*$ and thus any lift of a diagram with maps originating from $\mathcal{M}_*$ must preserve the basepoint as well. While there may be diagrams where the horizontal maps come from $\mathcal{M}$ may not produce a lift that preserves the basepoint $*$, as long as all the 4 maps in the diagram come from $\mathcal{M}_*$, the lift must preserve the basepoint.

The same argument allows us to show that fibrations have the right lifting property with respect to trivial cofibrations in $\mathcal{M}_*$.

Finally, we need to show the existence of two functorial factorizations that satisfy the required conditions.

We consider the factorization in $\mathcal{M}$ of a map $f : X \to Z$ that is in $\mathcal{M}_*$, i.e. $f = \beta(f) \circ \alpha(f)$ in $\mathcal{M}$, where $\alpha(f)$ is a cofibration and $\beta(f)$ a trivial fibration. If we show that $\alpha(f)$ and $\beta(f)$ preserve the basepoint, then we may conclude that (α, β) is a valid functorial factorization for the model structure of the category $\mathcal{M}_*$.

Let $\alpha(f) : X \to Y$ and $\beta(f) : Y \to Z$. Since (α, β) is a functorial factorization, we know that we have the following commutative diagram

$$\begin{tikzcd}
* \arrow[r, "\alpha(\mathrm{id}_*)"] \arrow[d] \arrow[rr, bend left, "\mathrm{id}"] & \tilde{Y} \arrow[r, "\beta(\mathrm{id}_*)"] \arrow[d, "g"] & * \arrow[d] \\
X \arrow[r, "\alpha(f)"'] \arrow[rr, bend right, "f"'] & Y \arrow[r, "\beta(f)"'] & Z
\end{tikzcd}$$

where g is a morphism $\tilde{Y} \to Y$ such that this diagram is commutative.

Therefore we have the following commutative triangles.

$$\begin{tikzcd}
 & * \arrow[dl] \arrow[dr, "g\alpha(\mathrm{id}_*)"] & \\
X \arrow[rr, "\alpha(f)"'] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & * \arrow[dl, "g\alpha(\mathrm{id}_*)"'] \arrow[dr] & \\
Y \arrow[rr, "\beta(f)"'] & & Z
\end{tikzcd}$$

This means that $\alpha(f)$ and $\beta(f)$ are morphisms in $\mathcal{M}_*$ and therefore there exists a functorial factorization (α, β) that sends $f : X \to Y$ of $\mathcal{M}_*$ to a cofibration $\alpha(f)$ followed by a trivial fibration $\beta(f)$ of $\mathcal{M}_*$.

The same reasoning applies to the functorial factorization (δ, γ) that sends $f : X \to Y$ of $\mathcal{M}_*$ to a trivial cofibration $\delta(f)$ followed by a fibration $\gamma(f)$. ■

Definition 3.8. Let $\mathcal{M}$ be a model category. We say that an object X of $\mathcal{M}$ is

- fibrant if the unique morphism from X to the terminal object $*$ is a fibration,
- cofibrant if the unique morphism from the initial object $\emptyset$ to X is a cofibration.

Proposition 3.9. Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}$ an object of $\mathcal{M}$. Then there exist a fibrant object RX along with a weak equivalence $X \to RX$ and a cofibrant object QX along with a weak equivalence $QX \to X$.

Proof. Let $X \in \mathcal{M}$. Since model categories have an initial object $\emptyset$ and a terminal object $*$ there exist two maps $\emptyset \to X$ and $X \to *$. Using the functorial factorization systems (a, b) and (c, d) we can factorize those two maps into $\emptyset \xrightarrow{f_1} Y \xrightarrow{g_1} X$ and $X \xrightarrow{f_2} Z \xrightarrow{g_2} *$, where f_1 is a cofibration, g_1 a trivial fibration, f_2 a trivial cofibration and g_2 a fibration.

Since g_1 is a trivial fibration, it is a weak equivalence and therefore Y is weakly equivalent to X and Y is a cofibrant object since $f_1 : \emptyset \to Y$ is a cofibration.

Likewise, since f_2 is a trivial cofibration, it is a weak equivalence and therefore Z is weakly equivalent to X and Z is a fibrant object since $g_2 : Z \to *$ is a fibration.

Thus, Y is the so called cofibrant replacement QX of X and Z is the fibrant replacement RX of X. ■

Remark. It follows directly from the proposition that we have a cofibrant replacement functor Q and a fibrant replacement functor R, given by $Q(X) = QX$ and $R(X) = RX$, where QX and RX are the cofibrant replacement of X and fibrant replacement of X respectively. The justification for Q and R being functors and not just maps of objects follows directly from the definition of RX and QX in the proof of Proposition 3.9. Since (a, b) and (c, d) are functorial factorizations we have that the following diagrams are commutative for any other morphism $f : X \to Y$

$$\begin{tikzcd}
\emptyset \arrow[r, "a(\emptyset\to X)"] \arrow[d, "\mathrm{id}_\emptyset"'] & QX \arrow[r, "b(\emptyset\to X)"] \arrow[d, "Q(f)"] & X \arrow[d, "f"] \\
\emptyset \arrow[r, "a(\emptyset\to Y)"'] & QY \arrow[r, "b(\emptyset\to Y)"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "c(X\to *)"] \arrow[d, "f"'] & RX \arrow[r, "d(X\to *)"] \arrow[d, "R(f)"] & * \arrow[d, "\mathrm{id}_*"] \\
Y \arrow[r, "c(Y\to *)"'] & RY \arrow[r, "d(Y\to *)"'] & *
\end{tikzcd}$$

where $Q(f)$ and $R(f)$ are the unique morphisms such that these two diagrams are commutative.

By uniqueness of $Q(f)$ and $R(f)$, $Q(\mathrm{id}_X) = \mathrm{id}_{QX}$ and $R(\mathrm{id}_X) = \mathrm{id}_{RX}$.

Composition is given by the fact that (a, b) and (b, c) are functorial, meaning that for two composable maps $gf : X \to Y \to Z$ we have $ab(gf) = a(gf)b(gf)$ and $cd(gf) = c(gf)d(gf)$. We have the following commutative diagrams.

$$\begin{tikzcd}
\emptyset \arrow[r, "a(\emptyset\to X)"] \arrow[d, "\mathrm{id}_\emptyset"'] & QX \arrow[r, "b(\emptyset\to X)"] \arrow[d, "Q(f)"] & X \arrow[d, "f"] \\
\emptyset \arrow[r, "a(\emptyset\to Y)"'] & QY \arrow[r, "b(\emptyset\to Y)"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\emptyset \arrow[r, "a(\emptyset\to Y)"] \arrow[d, "\mathrm{id}_\emptyset"'] & QY \arrow[r, "b(\emptyset\to Y)"] \arrow[d, "Q(g)"] & Y \arrow[d, "g"] \\
\emptyset \arrow[r, "a(\emptyset\to Z)"'] & QZ \arrow[r, "b(\emptyset\to Z)"'] & Z
\end{tikzcd}$$

It is clear that $gfb(\emptyset \to X) = gb(\emptyset \to Y)Q(f) = gb(\emptyset \to Y)Q(f) = b(\emptyset \to Z)Q(g)Q(f)$. Thus $Q(g)Q(f)$ is such that the following diagram is commutative.

$$\begin{tikzcd}
\emptyset \arrow[r, "a(\emptyset\to X)"] \arrow[d, "\mathrm{id}_\emptyset"'] & QX \arrow[r, "b(\emptyset\to X)"] \arrow[d, "Q(g)Q(f)"] & X \arrow[d, "gf"] \\
\emptyset \arrow[r, "a(\emptyset\to Z)"'] & QZ \arrow[r, "b(\emptyset\to Z)"'] & Z
\end{tikzcd}$$

By uniqueness of the map $QX \to QZ$ for which this diagram is commutative we conclude that $Q(gf) = Q(g)Q(f)$.

The same reasoning shows that $R(gf) = R(g)R(f)$. Therefore R and Q are indeed functors.

Theorem 3.10. Let $\mathcal{M}$ be a model category. A map f is a

- cofibration if and only if f has the left lifting property with respect to all trivial fibrations,
- trivial cofibration if and only if f has the left lifting property with respect to all fibrations,
- fibration if and only if f has the right lifting property with respect to all trivial cofibrations,
- trivial fibration if and only if f has the right lifting property with respect to all cofibrations.

Proof. Cofibrations are by definition in $llp(\mathcal{W} \cap \mathcal{F})$ and fibrations in $rlp(\mathcal{W} \cap \mathcal{C})$. We will first show that $llp(\mathcal{W} \cap \mathcal{F}) \subset \mathcal{C}$ and $rlp(\mathcal{W} \cap \mathcal{C}) \subset \mathcal{F}$.

Let $f : A \to B$ with $f \in llp(\mathcal{W} \cap \mathcal{F})$. We can factorize f into $g : A \to X$ and $h : X \to B$ where $g \in \mathcal{C}$ and $h \in \mathcal{F} \cap \mathcal{W}$. Since $f \in llp(\mathcal{W}\mathcal{F})$ there exists a lift

$j : B \to X$ as in the following diagram

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & X \arrow[d, "h"] \\
B \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "j"] & B
\end{tikzcd}$$

It is easy to see that f is a retract of g since we have the following diagram which exhibits g as a retract of f.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r, "\mathrm{id}"] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}"'] & X \arrow[r, "h"] & B
\end{tikzcd}$$

Since retracts of cofibrations are themselves cofibrations by definition, we know that f is a cofibration. Therefore $llp(\mathcal{W} \cap \mathcal{F}) \subset \mathcal{C}$.

Proving that $rlp(\mathcal{W} \cap \mathcal{C}) \subset \mathcal{F}$ can be done in a similar manner. Suppose $f : A \to B$ has the right lifting property with respect to all trivial cofibrations. Let $f = pi$ be a factorization such that i is a trivial cofibration and p a fibration. Since f has the right lifting property with respect to all trivial cofibrations, in particular with respect to i, we have the following diagram.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "i"'] & A \arrow[d, "f"] \\
Y \arrow[r, "p"'] \arrow[ur] & B
\end{tikzcd}$$

By Lemma 2.19, this diagram exhibits f as a retract of p. Since a retract of a fibration is also a fibration, we conclude that f is a fibration and therefore $rlp(\mathcal{W} \cap \mathcal{C}) \subset \mathcal{F}$.

The only remaining claim to prove is that a map is a trivial cofibration if and only if it has the left lifting property with respect to all fibrations and dually, a trivial fibration if and only if it has the right lifting property with respect to all cofibrations.

Suppose $f : X \to Y$ is a trivial cofibration. This implies that any fibration has the right lifting property with respect to f and therefore f has the left lifting property with respect to all fibrations.

Conversely, suppose f has the left lifting property with respect to all fibrations. Factor $f = pi$, where i is a trivial cofibration and p is a fibration, using one of the two functorial factorizations from the axioms for model categories.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & \tilde{Y} \arrow[d, "p"] \\
Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur] & Y
\end{tikzcd}$$

By Lemma 2.19, this diagram exhibits f as a retract of i since f has the left lifting property with respect to p. This implies that f is a trivial cofibration since a retract of a trivial cofibration is a trivial cofibration.

Suppose now that $f : X \to Y$ is a trivial fibration. This implies that any cofibration has the left lifting property with respect to f and therefore f has the right lifting property with respect to any cofibration.

Conversely, suppose f has the right lifting property with respect to any cofibration. Factor $f = pi$, where i is a cofibration and p is a trivial fibration.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
\tilde{Y} \arrow[r, "p"'] \arrow[ur] & Y
\end{tikzcd}$$

Relying once more on Lemma 2.19, we find that this diagram exhibits f as a retract of p because f has the right lifting property with respect to i. Since any retract of a trivial fibration is again a trivial fibration, we conclude that f must be a trivial fibration, as desired. ∎

Theorem 3.11. Let $\mathcal{M}$ be a model category. Then the class of cofibrations and the class of fibrations are closed under composition.

Proof. By Theorem 3.10 we know that we can express cofibrations as $llp(\mathcal{W} \cap \mathcal{F})$ and fibrations as $rlp(\mathcal{W} \cap \mathcal{C})$. By 2.25 we know that both those classes are closed under composition. Therefore the composition of two cofibrations or two fibrations yields a cofibration or fibration respectively. ∎

Lemma 3.12. Let $\mathcal{M}$ be a model category and $f : X \to Y$ a weak equivalence of $\mathcal{M}$. Then f factors through an object Z as an acyclic cofibration and a subsequent acyclic fibration.

Proof. Let $f : X \to Y$ be a weak equivalence of $\mathcal{M}$. Since we are in a model category $\mathcal{M}$ there exist two functorial factorizations. We can show the statement of the lemma using either one of the two.

One of the two factorizations gives us a factorization $f = pi$, where i is a cofibration and p is an acyclic fibration. Since f is a weak equivalence pi is as well and thanks to p being an acyclic fibration p is also a weak equivalence. By the 2-out-of-3 property that holds for weak equivalences in model categories this means that p, i and pi are weak equivalences.

Thus i is in fact an acyclic cofibration and not just a cofibration. ∎

Lemma 3.13. Let $\mathcal{M}$ be a model category and $f : X \to Y$ a weak equivalence between objects X and Y which are both cofibrant and fibrant objects of $\mathcal{M}$. Then there exists a factorization $X \to Z \to Y$ where $X \to Z$ is an acyclic cofibration, $Z \to Y$ an acyclic fibration and Z is a cofibrant and fibrant object.

Proof. The existence of an object Z such that the factorization of $X \to Y$ through Z consists of an acyclic cofibration followed by an acyclic fibration follows directly from Lemma 3.12.

To show that Z is a cofibrant and fibrant object, we only need to verify that $Z \to *$ is a fibration and that $\emptyset \to Z$ is a cofibration. Since X and Y are both cofibrant and fibrant, we know that $\emptyset \to X$ is a cofibration and $Y \to *$ is a fibration.

By Theorem 3.11 cofibrations and fibrations are closed under composition. Therefore $\emptyset \to X \to Z$ is a cofibration and $Z \to Y \to *$ is a fibration. Since $Z \to *$ and $\emptyset \to Z$ are unique morphisms, they must coincide with the cofibration and fibration and thus the unique morphisms $Z \to *$ and $\emptyset \to Z$ are a fibration and a cofibration respectively. Therefore Z is both a fibrant and cofibrant object. ■

Definition 3.14. Let $\mathcal{M}$ be a model category and X an object of $\mathcal{M}$. A *path object* for X, denoted as $\mathrm{Path}(X)$, is a factorization of the diagonal $\nabla_X : X \to X \times X$

$$\nabla_X : X \xrightarrow{i} \mathrm{Path}(X) \xrightarrow{(a,b)} X \times X,$$

where i is a weak equivalence.

If in addition $(a, b) : \mathrm{Path}(X) \to X \times X$ is a fibration, we say that $\mathrm{Path}(X)$ is a *good path object.*

Definition 3.15. Let $\mathcal{M}$ be a model category and X an object of $\mathcal{M}$. A *cylinder object* for X, denoted as $\mathrm{Cyl}(X)$, is a factorization of the codiagonal $\Delta_X : X \sqcup X \to X$

$$\Delta_X : X \sqcup X \xrightarrow{(a,b)} \mathrm{Cyl}(X) \xrightarrow{i} X,$$

where i is a weak equivalence.

If in addition $(a, b) : X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration, we say that $\mathrm{Cyl}(X)$ is a *good cylinder object.*

If (a, b) is a cofibration and i is an acyclic fibration we say that $\mathrm{Cyl}(X)$ is a *very good cylinder object.*

Remark. It is important to point out that the codiagonal and diagonal the above definitions are not related to the diagonal functor that we introduced in 2.14.

Lemma 3.16. Let $\mathcal{M}$ be a model category and let X be an object of $\mathcal{M}$. Then there exists a very good cylinder object $\mathrm{Cyl}(X)$.

Proof. Let $\Delta_X : X \sqcup X \to X$ be the codiagonal from the coproduct of X to X. Using one of the two functorial factorizations that come from the definition of a model category, we get a factorization $\Delta_X = pi$, where i is a cofibration and p is an acylic fibration. We get a very good cylinder object $\mathrm{Cyl}(X)$ in the form of the target of the map p or equivalently the source of the map i. ■

Definition 3.17. Let $\mathcal{M}$ be a model category and $f, g : X \to Y$ two morphisms of $\mathcal{M}$.

- A *left homotopy* $H_L : f \to g$ is a morphism $H_L : \mathrm{Cyl}(X) \to Y$ such that the following diagram is commutative.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & \mathrm{Cyl}(X) \arrow[d, "H_L"] & X \arrow[l] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

- A *right homotopy* $H_R : f \to g$ is a morphism $H_R : X \to \mathrm{Path}(Y)$ such that the following diagram is commutative.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "H_R"] \arrow[dr, "g"] & \\ Y & \mathrm{Path}(Y) \arrow[l] \arrow[r] & Y \end{tikzcd}$$

Definition 3.18. A model category $\mathcal{M}$ is cofibrantly generated if there exist two small sets I,J of morphisms of $\mathcal{M}$ such that

- $\mathrm{cof}(I) = \mathcal{C}$,
- $\mathrm{cof}(J) = \mathcal{W} \cap \mathcal{C}$,
- I and J allow for the small object argument, that is to say that the assumptions of the small object argument 2.37 are satisfied,

where $\mathcal{W}$ are the weak equivalences and $\mathcal{C}$ the cofibrations of the model structure of $\mathcal{M}$.

Remark. Since I and J are assumed to permit the small object argument in the definition given above, we can simplify $\mathrm{cof}(I)$ and $\mathrm{cof}(J)$ using the small object argument. It is easy to see that we have $\mathrm{cof}(I) = \mathrm{llp}(\mathrm{rlp}(I))$ and $\mathrm{cof}(J) = \mathrm{llp}(\mathrm{rlp}(J))$ and moreover we have $\mathcal{F} = \mathrm{rlp}(J)$ and $\mathcal{W} \cap \mathcal{F} = \mathrm{rlp}(I)$.

Definition 3.19. A model category $\mathcal{M}$ is said to be *combinatorial* if its underlying category is a locally presentable category (see Appendix 6.8) and its model structure is cofibrantly generated.

Lemma 3.20. Let $X \sqcup X \to \mathrm{Cyl}(X) \to X$ be a good cylinder object for some cofibrant object X of a model category $\mathcal{M}$. Then both components of the map $X \sqcup X \to \mathrm{Cyl}(X)$ are trivial cofibrations.

Dually, given a good path object $X \to \mathrm{Path}(X) \to X \times X$ for some fibrant object X, then both components of the map $\mathrm{Path}(X) \to X$ are trivial fibrations.

Proof. We have two inclusions of the form $i_0, i_1 : X \to X \sqcup X \to \mathrm{Cyl}(X)$. These are clearly cofibrations since they can be obtained as the pushout of the cofibration $\emptyset \to X$. To be more precise, suppose we have a trivial fibration $f : A \to B$ and the following commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "j"'] & A \arrow[d, "f"] \\ X \sqcup X \arrow[r] & B \end{tikzcd}$$

Adding the cofibration $\emptyset \to X$ to the commutative square as follows

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "i"'] & X \arrow[r] \arrow[d, "j"] & A \arrow[d, "f"] \\ X \arrow[r] & X \sqcup X \arrow[r] & B \end{tikzcd}$$

where i is a cofibration since X is a cofibrant object of $\mathcal{M}$, allows us to find a lift $h : X \to A$ as in the following diagram.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "i"'] & X \arrow[r] \arrow[d, "j"] & A \arrow[d, "f"] \\ X \arrow[r] \arrow[urr, dashed, bend right, "h"] & X \sqcup X \arrow[r] & B \end{tikzcd}$$

By the universal property of the pushout we therefore have the following commutative diagram:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "i"'] & X \arrow[r] \arrow[d, "j"] & A \arrow[d, "f"] \\
X \arrow[r] \arrow[urr, dashed, bend left=10, "h" near end] & X \sqcup X \arrow[r] \arrow[ur, dashed, bend right, "\tilde{h}"'] & B
\end{tikzcd}$$

This diagram clearly shows that the commutative square consisting of $j : X \to X \sqcup X$ and $f : A \to B$ from above has a lift in the form of $\tilde{h} : X \sqcup X \to A$. Therefore j has the left lifting property with respect to all trivial fibrations and therefore j is a cofibration by 3.10.

This implies that both natural inclusions $X \to X \sqcup X$ are cofibrations. Since $X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration so are the two maps $i_0, i_1 : X \to \mathrm{Cyl}(X)$ since each of them is a composition of one of the two natural inclusions $X \to X \sqcup X$ and the cofibration $X \sqcup X \to \mathrm{Cyl}(X)$.

The fact that they are also weak equivalences follows from the 2-out-of-3 property. Clearly $id_X : X \to X \sqcup X \to \mathrm{Cyl}(X) \to X$ and $\mathrm{Cyl}(X) \to X$ are weak equivalences. By the 2-out-of-3 property that holds for model categories this means that id_X, $\mathrm{Cyl}(X) \to X$, i_0 and i_1 are all weak equivalences.

The second part of the lemma is formally dual. ■

Lemma 3.21. Let $\mathcal{M}$ be a model category and $\eta : f \Rightarrow_L g : X \to Y$ a left homotopy, where Y is a fibrant object. Then for any good cylinder object $\widetilde{\mathrm{Cyl}(X)}$ for X there exists a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "f"'] & \widetilde{\mathrm{Cyl}(X)} \arrow[d, "\tilde{\eta}"] & X \arrow[l] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

Dually, if $\eta : f \Rightarrow_R g : X \to Y$ is a right homotopy, where X is a cofibrant object, then for any good path object $\widetilde{\mathrm{Path}(X)}$ for X, there is a commutative diagram of the form

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[d, "\tilde{\eta}"] \arrow[dr, "g"] & \\
Y & \widetilde{\mathrm{Path}(X)} \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

Proof. Let $\eta : \mathrm{Cyl}(X) \to Y$ be a given left homotopy. Using the factorization from the definition of a model category, we can factor η as $\eta : \mathrm{Cyl}(X) \overset{\in \mathcal{C}}{\to} Z \overset{\in \mathcal{W} \cap \mathcal{F}}{\to} Y$.

We have the following two diagrams

$$\begin{tikzcd}
X \sqcup X \arrow[r] \arrow[d, "\in \mathcal{C}"'] & \mathrm{Cyl}(X) \arrow[r, "\in \mathcal{C}"] & Z \arrow[d, "\in \mathcal{W} \cap \mathcal{F}"] \\
\widetilde{\mathrm{Cyl}(X)} \arrow[rr] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Cyl}(X) \arrow[r, "\eta"] \arrow[d, "\in \mathcal{C}"'] & Y \arrow[d, "\in \mathcal{W} \cap \mathcal{F}"] \\
Z \arrow[r] & *
\end{tikzcd}$$

Since cofibrations have the left-lifting property with respect to acyclic fibrations, we can find liftings

$$\begin{tikzcd}
X \sqcup X \arrow[r] \arrow[d, "\in \mathcal{C}"'] & \mathrm{Cyl}(X) \arrow[r, "\in \mathcal{C}"] & Z \arrow[d, "\in \mathcal{W} \cap \mathcal{F}"] \\
\widetilde{\mathrm{Cyl}(X)} \arrow[rr] \arrow[urr, "\psi"] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Cyl}(X) \arrow[r, "\eta"] \arrow[d, "\in \mathcal{C}"'] & Y \arrow[d, "\in \mathcal{W} \cap \mathcal{F}"] \\
Z \arrow[r] \arrow[ur, "\varphi"] & *
\end{tikzcd}$$

We claim that $\tilde{\eta} := \varphi \circ \psi$ yields the desired left homotopy for $\widetilde{\mathrm{Cyl}(X)}$. We have

$$\begin{tikzcd}
& \mathrm{Cyl}(X) \arrow[d] & & \\
X \arrow[ur] \arrow[rr, bend left, "f"] \arrow[dr] & Z \arrow[r, "\varphi"] & Y & X \arrow[l, "g"'] \arrow[ull] \arrow[dll] \\
& \widetilde{\mathrm{Cyl}(X)} \arrow[u, "\psi"] & &
\end{tikzcd}$$

From this diagram we extract the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "f"'] & \widetilde{\mathrm{Cyl}(X)} \arrow[d, "\tilde{\eta}"] & X \arrow[l] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

which is what we wanted to show.

The second statement in the lemma is formally dual. ■

Lemma 3.22. Let $\mathcal{M}$ be a model category and $f, g : X \to Y$ two parallel morphisms.

- If X is cofibrant, then the existence of a left homotopy $f \Rightarrow_L g$ implies the existence of a right homotopy $f \Rightarrow_R g$ for any good path object.

- If Y is fibrant, then the existence of a right homotopy $f \Rightarrow_R g$ implies the existence of a left homotopy $f \Rightarrow_L g$ for any good cylinder object.

Proof. Let $\eta : \mathrm{Cyl}(X) \to Y$ be the given left-homotopy. Using 3.21 we assume that $\mathrm{Cyl}(X)$ is a good cylinder object, otherwise we replace it. By 3.20 we have a lift in the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\mathcal{W} \cap \mathcal{C} \ni"'] & \mathrm{Path}(Y) \arrow[d, "\in \mathcal{F}"] \\
\mathrm{Cyl}(X) \arrow[r] & Y \times Y
\end{tikzcd}$$

since fibrations have the right lifting property with respect to acyclic cofibrations. This means that there exists $h : \mathrm{Cyl}(X) \to \mathrm{Path}(Y)$ such that we get a commutative diagram.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\mathcal{W}\cap\mathcal{C}\ni"'] & \mathrm{Path}(Y) \arrow[d, "\in\mathcal{F}"] \\
\mathrm{Cyl}(X) \arrow[r] \arrow[ur, "h"] & Y \times Y
\end{tikzcd}$$

It is easy to see that $\eta := h \circ i$ is the desired right homotopy. ■

Proposition 3.23. Let $\mathcal{M}$ be a model category, X a cofibrant object and Y a fibrant object. Then the existence of a left-homotopy $f \Rightarrow_L g$ and right homotopy $f \Rightarrow_R g$ coincide and form equivalence relations for the hom set $\mathrm{Hom}(X, Y)$.

Proof. The fact that the existence of a left-homotopy coincides with the existence of a right-homotopy is a direct consequence of 3.22.

To show that we have an equivalence relation for the hom set, we first remark that symmetry and reflexivity are obvious. We only have to show transitivity.

Suppose we have two left homotopies $f \Rightarrow_L g : X \to Y$ and $g \Rightarrow_L h : X \to Y$. Using 3.21 we can exhibit these left homotopies in the form of two commutative diagrams:

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(X) \arrow[d, "\psi"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[dr, "g"'] & \mathrm{Cyl}(X) \arrow[d, "\varphi"] & X \arrow[l, "i_1"'] \arrow[dl, "h"] \\
& Y &
\end{tikzcd}$$

We now consider the following diagram

$$\begin{tikzcd}
& & & X \arrow[d, "i_0"] & \\
& X \arrow[rr, "i_1"] \arrow[d, "i_0"] & & \mathrm{Cyl}(X) \arrow[d] \arrow[ddr] & \\
X \arrow[r, "i_1"] & \mathrm{Cyl}(X) \arrow[rr] \arrow[drrr] & & \mathrm{Cyl}(X) \sqcup \mathrm{Cyl}(X) \arrow[dr] & \\
& & & & X
\end{tikzcd}$$

which we obtain by considering the pushout to insert $\mathrm{Cyl}(X) \sqcup \mathrm{Cyl}(X)$. This implies that $\mathrm{Cyl}(X) \sqcup \mathrm{Cyl}(X)$ is a cylinder object and we naturally have the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr] \arrow[ddrr, bend right, "f"'] & & \mathrm{Cyl}(X) \sqcup \mathrm{Cyl}(X) \arrow[dd, "(\varphi{,}\psi)"] & & X \arrow[ll] \arrow[ddll, bend left, "h"] \\
& X \arrow[ur] \arrow[dr, "g"] & & X \arrow[ul] \arrow[dl, "g"'] & \\
& & Y & &
\end{tikzcd}$$

which shows that $(\varphi, \psi) : \mathrm{Cyl}(X) \sqcup \mathrm{Cyl}(X) \to Y$ is a left homotopy $f \Rightarrow_L h$ as desired. This means that the left-homotopy relation is transitive. ■

We will now present a few simple examples of model categories before discussing model structures for the category **Top** of topological spaces.

Example 3.24. Let $\mathcal{M}$ be a bicomplete category. We can impose three distinct model structures on $\mathcal{M}$ by choosing one of the three distinguished classes of morphism $(\mathcal{W}, \mathcal{F}, \mathcal{C})$ as the class of isomorphisms of $\mathcal{M}$ and the other two classes to be all morphisms. As an example, let $f : X \to Y$ be a weak equivalence if and only if it is an isomorphism and let any morphism in $\mathcal{M}$ be a fibration and a cofibration. Suppose that f and g are morphisms of $\mathcal{M}$ and gf is defined and suppose that 2-out-of-3 of f, g and gf are weak equivalences, i.e. isomorphisms, then so is the third.

In fact, suppose $f : X \to Y$ and $g : Y \to Z$ are isomorphisms, then so is $gf : X \to Z$ trivially.

If f and gf are isomorphisms, then $f : X \to Y$ has an inverse $h : Y \to X$ such that $fh = 1_Y$ and $hf = 1_X$ and $gf : X \to Z$ has an inverse $l : Z \to X$ such that $gfl = 1_Z$ and $lgf = 1_X$. This means that fl is the inverse of g since $gfl = 1_Z$ and $flg = flgfh = f1_Xh = fh = 1_Y$, thus g is an isomorphism.

If g and gf are isomorphisms, then g has an inverse h and gf has an inverse l and we have $gh = 1_Z$, $hg = 1_Y$, $gfl = 1_Z$ and $lgf = 1_X$. It is trivial to see that f has an inverse, specifically the inverse of f is given by lg and we have $lgf = 1_X$ and $flg = hgflg = h1_Zg = hg = 1_Y$.

Therefore the 2-out-of-3 property holds.

Suppose f is a retract of g. Since all morphisms are fibrations and cofibrations, we only need to check that if g is a weak equivalence, i.e. an isomorphism, then so is f. This is easy to verify. Since f is a retract of g we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & C \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r, "j"'] \arrow[rr, bend right, "1_B"'] & D \arrow[r, "p"'] & B
\end{tikzcd}$$

We need to show that f has an inverse. Since g is an isomorphism, there exists a morphism $h : D \to C$ such that $gh = 1_D$ and $hg = 1_C$. Therefore we have $1_A = ri = rhgi = rhjf$ and $1_B = pj = pghj = frhj$ and thus we see that rhj is the inverse of f and therefore f is an isomorphism and thus a weak equivalence.

To verify that cofibrations have the left lifting property with respect to morphisms in $\mathcal{W} \cap \mathcal{F}$ and fibrations have the right lifting property with respect to morphisms in $\mathcal{W} \cap \mathcal{C}$ we remark that $\mathcal{W} \cap \mathcal{F} = \mathcal{W} \cap \mathcal{C} = \mathcal{W}$, i.e. trivial fibrations and trivial cofibrations are exactly the weak equivalences. So we need to verify that cofibrations have the left lifting property with respect to isomorphisms and fibrations have the right lifting property with respect to isomorphisms.

Let $f : X \to Y$ be an isomorphisms, $g : A \to B$ a cofibration and $h : C \to D$ a fibration.

Suppose we have the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "g"'] & X \arrow[r, "\tilde{\varphi}"] \arrow[d, "f"'] & C \arrow[d, "h"] \\ B \arrow[r, "\psi"'] & Y \arrow[r, "\tilde{\psi}"'] & D \end{tikzcd}$$

Since f is an isomorphism there must be an inverse, say $l : Y \to X$ such that $fl = 1_Y$ and $lf = 1_X$.

Is is easy to see that $l\psi : B \to X$ and $\tilde{\varphi}l : B \to D$ are such that the following diagram is commutative.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "g"'] & X \arrow[r, "\tilde{\varphi}"] \arrow[d, "f"'] & C \arrow[d, "h"] \\ B \arrow[r, "\psi"'] \arrow[ur, "l\psi"] & Y \arrow[r, "\tilde{\psi}"'] \arrow[ur, "\tilde{\varphi}l"] & D \end{tikzcd}$$

Lastly, we need to define two functorial factorizations (a, b) and (c, d) on $\mathcal{M}$ such that for any morphism $f : X \to Y$ we have $af \in \mathcal{C}$, $bf \in \mathcal{W} \cap \mathcal{F}$, $cf \in \mathcal{W} \cap \mathcal{C}$ and $df \in \mathcal{F}$.

We define a and d to be the identity functors and $b(f)$ to be the identity of the codomain of f and $c(f)$ to be the identity of the domain of f.

It is trivial to see that $af \in \mathcal{C}$ and $df \in \mathcal{F}$, since $\mathcal{F} = \mathcal{C}$ are all morphisms of $\mathcal{M}$. Since $b(f) = 1_Y$ it is clearly an isomorphism and thus in $\mathcal{W} \cap \mathcal{F} = \mathcal{W}$ and the same holds for $cf = 1_X \in \mathcal{W} \cap \mathcal{C}$. We will leave it to the reader to verify that as defined (a, b) and (c, d) are indeed functorial factorizations.

Example 3.25. Given two model categories $\mathcal{M}_1$ and $\mathcal{M}_2$ there exists a natural model structure for the product category $\mathcal{M}_1 \times \mathcal{M}_2$, given by $(\mathcal{W}_1 \times \mathcal{W}_2, \mathcal{F}_1 \times \mathcal{F}_2, \mathcal{C}_1 \times \mathcal{C}_2)$, where $(\mathcal{W}_1, \mathcal{F}_1, \mathcal{C}_1)$ and $(\mathcal{W}_2, \mathcal{F}_2, \mathcal{C}_2)$ are the model structures of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. This is the so called product model structure.

Remark. From example 3.24 it is clear that for a given category we can find multiple model structures that are not necessarily equivalent. For now we will not clarify what we mean by two model structures being equivalent since there are multiple ways in which model structures can be equivalent. We will discuss various notions of equivalence between model structures at a later point, in particular when we introduce the notion of Quillen equivalence.

§3.2 The homotopy category of a model category

Model categories provide a natural context in which to carry out homotopy theory. We will now make this statement more concrete and obvious to the uninitiated by introducing the notion of the homotopy category of a model category. While we will introduce the notion of a homotopy category for categories endowed with strictly weaker structures than model structures in a subsequent section, doing so will be much more abstract. One of the biggest advantages that we have when working with model categories is that we can give a relatively explicit construction of the homotopy category of a model category.

Definition 3.26. Let $\mathcal{M}$ be a model category with weak equivalences $\mathcal{W}$, cofibrations $\mathcal{C}$ and fibrations $\mathcal{F}$. The homotopy category of $\mathcal{M}$, denoted by $\mathrm{Ho}\mathcal{M}$ is the category whose

- objects are given by objects that are both fibrant and cofibrant in the model category $\mathcal{M}$,
- morphisms are homotopy classes of morphisms of $\mathcal{M}$, i.e. equivalences classes of morphisms under left homotopy.

Remark. Since Ho$\mathcal{M}$ is unique up to equivalence of categories, we speak of the homotopy category of the model category $\mathcal{M}$.

Once we introduce the notion of homotopy categories for categories with weaker structures than that of a model structure, we will not be able to provide as concrete of a description of the homotopy category of a category and instead will define the homotopy category using a universal property. Nonetheless, the reader may already get a rather good glimpse of the universal property of the homotopy category as a result of the following result.

Theorem 3.27. Let $\mathcal{M}$ be a model category. A weak equivalence between two objects which are fibrant and cofibrant is a homotopy equivalence, that is turns into an isomorphism in Ho$\mathcal{M}$.

Proof. Let $f : X \to Y$ be a weak equivalence in $\mathcal{M}$ and suppose that X and Y are both fibrant and cofibrant. By lemma 3.12, we know that f factors through some object Z as the composition of an acyclic cofibration and an acyclic fibration. By lemma 3.13 X and Y are each fibrant and cofibrant, this must also be true for Z. This means that our job is much easier since it means that we only have to show that acyclic fibrations and acyclic cofibrations between objects that are both fibrant and cofibrant are homotopy equivalences.

Using above reasoning, suppose that $f : X \to Y$ is an acyclic fibration with X and Y each being fibrant and cofibrant. We summarize in the form of the following commutative diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y \arrow[r, dotted, leftrightarrow, "1_Y"'] & Y
\end{tikzcd}$$

where g is a cofibration since Y is cofibrant.

By definition of model categories, cofibrations have the left lifting property with respect to acyclic fibrations and therefore g has the left lifting property with respect to f, which means there exists a morphism $r : Y \to X$ such that the following diagram is commutative.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y \arrow[ur, "r"] \arrow[r, dotted, leftrightarrow, "1_Y"'] & Y
\end{tikzcd}$$

Clearly r is a right inverse of f since $fr = 1_Y$. We need to show that r is a left inverse of f up to left homotopy. Let Cyl(X) be any very good cylinder object on X, whose existence of follows from lemma 3.16. This means we have a factorization

$X \sqcup X \xrightarrow{i} \mathrm{Cyl}(X) \xrightarrow{p} X$, where i is an cofibration and p is an acyclic fibration and we have the following diagram

$$\begin{tikzcd}
X \sqcup X \arrow[r, "{(rf,1_X)}"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
\mathrm{Cyl}(X) \arrow[r, "fp"'] & Y
\end{tikzcd}$$

where $i \in \mathcal{C}$ and $p \in \mathcal{W} \cap \mathcal{F}$.

This diagram is commutative since r is a right inverse of f. By this we mean that in the above diagram, if we compose $(rf, 1_X)$ with f we end up with a map $(frf, f1_X) : X \sqcup X \to Y$ and since r is a right inverse of f this is the same as the map $(f, f1_X) = (f, f)$ and since ∇ is the codiagonal with factorization through $\mathrm{Cyl}(X)$ via maps $i : X \sqcup X \to \mathrm{Cyl}(X)$ and $p : \mathrm{Cyl}(X) \to X$ we have $fpi : X \sqcup X \to Y$ with $fpi = f\nabla = f(1_X, 1_X) = (f, f)$ and thus the diagram is indeed commutative. Since cofibrations have the left lifting property with respect to acyclic fibrations, we know that there exists a lift $h : \mathrm{Cyl}(X) \to X$ such that $(rf, 1_X) = hi$ and $fp = fh$. Clearly this implies that we have the following commutative diagram, which exhibits h as a left homotopy between 1_X and rf and thus r is a left inverse of f up to left homotopy, which means that f is indeed a homotopy equivalence.

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[dr, "rf"'] & \mathrm{Cyl}(X) \arrow[d, "h"] & X \arrow[l, "i_2"'] \arrow[dl, "1_X"] \\
& X &
\end{tikzcd}$$

■

Proposition 3.28. Let $\mathcal{M}$ be a model category. Denote by $\mathcal{M}_c, \mathcal{M}_f, \mathcal{M}_{cf}$ the full subcategories of cofibrant, fibrant, cofibrant and fibrant objects of $\mathcal{M}$. The inclusion functors $\mathcal{M}_{cf} \to \mathcal{M}_c \to \mathcal{M}$ and $\mathcal{M}_{cf} \to \mathcal{M}_f \to \mathcal{M}$ induce equivalences of the homotopy categories of all subcategories and the model category $\mathcal{M}$.

Proof. Let $\mathcal{M}_c \xrightarrow{i} \mathcal{M}$ be the inclusion of $\mathcal{M}_c$ into $\mathcal{M}$. The map i clearly preserves weak equivalences, since any weak equivalence between two cofibrant objects is a weak equivalence in $\mathcal{M}$ as well. This means that i induces a functor $\mathrm{Ho}(i) : \mathrm{Ho}(\mathcal{M}_c) \to \mathrm{Ho}(\mathcal{M})$.

We need to show that there exists an inverse functor of $\mathrm{Ho}(i)$. We claim that the cofibrant replacement functor Q acts as the inverse of $\mathrm{Ho}(i)$. For any object X, the cofibrant replacement functor Q yields a cofibrant object QX. Moreover, we have a natural trivial fibration $QX \overset{b(\emptyset \to X)}{\to} X$.

It is obvious that Q preserves weak equivalences by the 2-out-of-3 property for weak equivalences, since the functorial factorization of $\emptyset \to X$ and $\emptyset \to Y$ yields trivial fibrations $b(\emptyset \to X) : QX \to X$ and $b(\emptyset \to Y) : QY \to Y$ that commute with any weak equivalence $f : X \to Y$, that is $fb(\emptyset \to X) = b(\emptyset \to Y)Q(f)$ and therefore $b(\emptyset \to Y)Q(f)$ is a weak equivalence since $b(\emptyset \to X)$ and f are. Using the 2-out-of-3 property for weak equivalences one more time, we conclude that $Q(f)$ is a weak equivalence since $b(\emptyset \to Y)$ and $b(\emptyset \to Y)Q(f)$ are weak equivalences. Therefore if $f : X \to Y$ is a weak equivalence, then so is $Q(f)$ and therefore Q preserves weak equivalences.

Therefore, Q induces a functor $\mathrm{Ho}(Q) : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M}_c)$ and both Q and i are so called homotopical functors.

Since the maps $QX \to X$ and $X \to QX$ are weak equivalences and commute appropriately with any other map $f : X \to Y$, we have natural transformations $Q \circ i \to 1_{\mathcal{M}_c}$ and $i \circ Q \to 1_{\mathcal{M}}$ and those two maps are natural weak equivalences. This natural transformation induces two natural transformations $\mathrm{Ho}(i) \circ \mathrm{Ho}(Q) \to 1_{\mathrm{Ho}(\mathcal{M}_c)}$ and $\mathrm{Ho}(Q) \circ \mathrm{Ho}(Q) \to 1_{\mathrm{Ho}(\mathcal{M})}$. Since on the component level the natural transformation is always a weak equivalence, the induced natural transformations must be isomorphisms on the component level, which shows that we have natural isomorphisms $\mathrm{Ho}(i) \circ \mathrm{Ho}(Q) \to 1_{\mathrm{Ho}(\mathcal{M}_c)}$ and $\mathrm{Ho}(Q) \circ \mathrm{Ho}(Q) \to 1_{\mathrm{Ho}(\mathcal{M})}$ and thus $\mathrm{Ho}(Q)$ and $\mathrm{Ho}(i)$ form an equivalence of categories and act as each others inverse.

Similarly, we can show that $\mathrm{Ho}(\mathcal{M}_f) \to \mathrm{Ho}(\mathcal{M})$ is an equivalence of categories. The argument is completely dual to what we already showed, instead relying on the fibrant replacement functor R and the inclusion.

Once the equivalences $\mathrm{Ho}(\mathcal{M}_f) \to \mathrm{Ho}(\mathcal{M})$ and $\mathrm{Ho}(\mathcal{M}_c) \to \mathrm{Ho}(\mathcal{M})$ are shown, the remaining equivalences follow directly, since the inclusion of both fibrant and cofibrant objects into the cofibrant or fibrant subcategory respectively corresponds to the already settled cases. ∎

§3.3 Examples of model categories

3.3.1 Model structure on topological spaces

We have previously noted that model categories present a natural setting in which to carry out homotopy theory. For this to be true in a meaningful sense there must be a way to reconcile classical homotopy theory of topological spaces and the more abstract setting of model categories. As we shall see, this is entirely possible by way of introducing a model structure on the category of topological spaces that results in a homotopy theory that coincides with our classical understanding of homotopy theory. We will introduce two distinct model structures for topological spaces.

As we remarked before, a category can potentially be given multiple model structures that are not necessarily equivalent to each other. This is indeed the case for the category Top of topological spaces.

Example 3.29. Let Top be the category of topological spaces. A map $f : X \to Y$ is

- a weak equivalence if f is a weak homotopy equivalence,
- a fibration if f is a Serre fibration,
- a cofibration if f is a retract of some map $X \to Z$ where Z is constructed by attaching cells to X.

The category Top along with this model structure yields a homotopy category $\mathrm{Ho}(\mathsf{Top})$ that is equivalent to the standard homotopy category of CW-complexes.

Every object is fibrant, cofibrant objects are given by spaces which are retracts of generalized CW complexes, where generalized means that we do not require cells to be attached in order of their dimension.

Remark. The model structure for the category of topological spaces in the example above defines weak equivalences as weak homotopy equivalences. One may wonder whether it is also possible to define weak equivalences to be ordinary homotopy equivalences. As we will see in the following example, this is indeed possible.

Example 3.30. Let Top be the category of topological spaces. A map $f : X \to Y$ is

- a weak equivalence if f is a homotopy equivalence,
- a fibration if f is a Hurewicz fibration,
- a cofibration if f is a closed Hurewicz cofibration.

The category Top along with this model structure yields a homotopy category $\mathrm{Ho}(\mathsf{Top})$ that is equivalent to the standard homotopy category of topological spaces.

3.3.2 Model structure for modules

We will now discuss model structures for the category of modules over a ring R. This is primarily motivated by the example of not Quillen equivalent model categories that we will discuss later on, since both model categories that we will be looking at are categories of modules over two particular rings. In this part we will closely follow Hovey [13], Section 2.2. This is also where proofs for some results presented in this section may be found.

We first recall some basic definitions concerning rings and modules.

Definition 3.31. Let R be a ring. An R-module Q is called an *injective module* if any short exact sequence

$$0 \to Q \to N \to M \to 0$$

of R-modules is a split exact sequence.

Similarly, an R-module P is called a *projective module* if any short exact sequence

$$0 \to N \to M \to P \to 0$$

of R-modules is a split exact sequence.

Definition 3.32. Let R be a ring and $f, g : M \to N$ two maps between R-modules N and M. We say that f and g are stably equivalent if $f - g$ factors through a projective module, that is to say we can find a factorization of $f - g : M \to P \to N$ where P is a projective module.

Stable equivalence is an equivalence relation.

Lemma 3.33. As defined above, the notion of stable equivalence holds under composition. By this we mean that if $f \sim g$ are stably equivalent and h and k are maps such that hf, hg, fk, gk are defined, then $f \sim g$ implies that $hf \sim hg$ and $fk \sim gk$.

Proof. Let $f, g : M \to N$ be stably equivalent and let $h : N \to O$ and $k : Q \to N$ be maps between R-modules.

Since $f \sim g$, there exists a factorization of $f - g = pi$ through some projective module P, where $i : M \to P$ and $p : P \to N$. Since we are working with maps between R-modules we have $hf - hg = h(f - g)$. Therefore $h(f - g) = hpi = (hp)i$ and therefore $hf - hg$ factors through the same projective module P as $f - g$. ∎

Definition 3.34. Let R be a ring. We define the *stable category of R-modules* as the category whose objects are left R-modules and morphisms are given by stable equivalence classes of maps between R-modules.

Definition 3.35. A ring R is called a Frobenius ring if injective and projective modules coincide, that is to say that every injective module is a projective module and vice-versa.

It turns out that if R is a Frobenius ring, then the stable category of R-modules happens to be homotopy category of a particular model structure placed on top of the category of R-modules. In fact this particular model structure turns out to be cofibrantly generated.

Definition 3.36. Let R be a Frobenius ring. Let I be the set of inclusions of left ideals of R into R and J the set of inclusions of the zero ideal into R.

We say that a map f of R-modules is a fibration if it has the right lifting property with respect to J and a cofibration if $f \in \mathrm{cof}(I)$ (see 2.35).

Lemma 3.37. Let R be a Frobenius ring and let $f : M \to N$ be a map of R-modules. Then

- f is a fibration if and only if it is surjective,
- f is a trivial fibration if and only if p is a surjection and its kernel is a projective module.

Proposition 3.38. Let R be a Frobenius ring. A map of R-modules is in $\mathrm{inj}(I)$ if and only if it is a surjection with injective kernel. Moreover, $\mathrm{inj}(I)$ is given exactly by trivial fibrations.

Lemma 3.39. Let R be a Frobenius ring. A map of R-modules is in $\mathrm{cof}(I)$ if and only if it is an injection.

Lemma 3.40. Let R be a Frobenius ring. A map of R-modules is in $\mathrm{cof}(J)$ if and only if it is an injection with projective cokernel. Moreover, all maps in $\mathrm{cof}(J)$ are stable equivalences.

Theorem 3.41. Let R be a Frobenius ring. There exists a cofibrantly generated model structure on the category of R-modules with cofibrations given by injections, fibrations by surjections and weak equivalences by stable equivalences.

We call this model category the *stable module category of the ring R* or the model category of stable modules over R and denote it by $\mathrm{Stmod}(R)$.

Proposition 3.42. The cofibrantly generated model structure on the category of R-modules has the following properties:

- Every object is cofibrant and fibrant.
- Two maps f, g between R-modules are stably equivalent if and only if they are left or right homotopic in the cofibrantly generated model structure.

§3.4 Weaker structures than model categories

Many interesting properties and results surrounding model categories show up in one way or another when we do not require a category to have a model structure but instead a weaker and more general structure. We will introduce two notions that are weaker than the notion of a model structure in the form of homotopical categories and categories with weak equivalences. In many ways model categories are important precisely because they provide a more tangible and concrete context than for instance $(\infty, 1)$-categories.

Definition 3.43. A *homotopical category* is a category with a distinguished class of morphisms, the so called weak equivalences, such that

- every identity map is a weak equivalence,
- weak equivalences have the 2-out-of-6 property: For any triple of morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ it holds that if hg and gf are weak equivalences, then so are f, g, h and hgf.

Remark. Every model category is trivially a homotopical category.

Definition 3.44. A *category with weak equivalences* is a category C along with a subcategory $\mathcal{W} \subset C$ such that

1. $\mathcal{W}$ contains all isomorphisms of C,
2. $\mathcal{W}$ satisfies the 2-out-of-3 property: for any two composable morphisms f, g of C, if two out of f, g, gf are in $\mathcal{W}$, then all of them are.

Remark. Every homotopical category is a category with weak equivalences. This follows from the fact that the 2-out-of-6 property along with all identity maps being weak equivalences implies the 2-out-of-3 property.

In our introduction of the notion of the homotopy category of a model category we justified the claim that model categories provide a natural context in which to carry out homotopy theory. We will now quickly touch on how we can associate a homotopy category to weaker structures than model structures. This will also provide a very good argument in favor of working with the stronger model structure instead of weaker structures introduced in the previous section.

Definition 3.45. Let $\mathcal{M}$ be a category with weak equivalences $\mathcal{W}$. The *"homotopy category"* Ho$\mathcal{M}$ is constructed by adding inverses for all weak equivalences. More precisely, we consider the free category generated by $\mathcal{M}$ and formal inverses w^{-1} for each $w \in \mathcal{W}$, where $w^{-1} : Y \to X$ for $w : X \to Y$, whose objects coincide with objects of $\mathcal{M}$ and morphisms are given by finite strings $(f_1, \ldots, f_n)$ of morphisms such that the morphisms can be composed and each morphism in a given string is either a morphism of $\mathcal{M}$ or a formal inverse w^{-1} for some $w \in \mathcal{W}$. The composition in this free category is given by concatenation of strings and the identity element at a given object is given by an empty string. Ho$\mathcal{M}$ is obtained as a quotient category of this free category under the equivalence relations

- for any $X \in \mathcal{M}$ we have $1_X = (1_X)$,
- for morphisms f, g of $\mathcal{M}$ that are composable we have $(f, g) = (gf)$,
- for any weak equivalence $w : X \to Y$ we have $(w, w^{-1}) = 1_X$ and $(w^{-1}, w) = 1_Y$.

Remark. At this point it is extremely important to remark that as defined above, Ho$\mathcal{M}$ is not necessarily a category. In fact, requiring that $\mathcal{M}$ is a model category and not just a category with a choice of weak equivalences guarantees that Ho$\mathcal{M}$ is an actual category. For a full proof of this result the reader may consult [13], specifically Theorem 1.2.10.

Proposition 3.46. Let $\mathcal{M}$ be a category with weak equivalences. If the homotopy category Ho$\mathcal{M}$ exists, then it is the unique, up to equivalence of categories, category which is universal with respect to the existences of a functor $Q : \mathcal{M} \to$ Ho$\mathcal{M}$ that sends every weak equivalence in $\mathcal{M}$ to an isomorphism in Ho$\mathcal{M}$.

Remark. It is unfortunate that many authors denote the homotopy category of a category $\mathcal{M}$ by Ho$\mathcal{M}$, since the homotopy category of $\mathcal{M}$ depends on the choice of weak equivalences. Sometimes less ambiguous notations are used, such as $\mathcal{W}^{-1}\mathcal{M}$ or $\mathcal{M}[\mathcal{W}^{-1}]$, which highlights the fact that the homotopy category of a category $\mathcal{M}$ with weak equivalences $\mathcal{W}$ is in fact the localization of $\mathcal{M}$ at the weak equivalences $\mathcal{W}$.

§4 Quillen Equivalence

Since model categories are categories we naturally have a notion of equivalence of two model categories in the form of the categorical equivalence. However, this is not the correct notion of equivalence for model categories. This is where the notion of Quillen equivalence comes in. Before we define what a Quillen equivalence is we will define Quillen functors and introduce the notion of derived functors. After the relevant definitions we will attempt to provide a concise and coherent exposition on the relevance of Quillen equivalence.

§4.1 Definitions and basic results

Definition 4.1. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two model categories. A pair of adjoint functors

$$(L, R) : \mathcal{M}_1 \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{M}_2,$$

where L is the left adjoint and R the right adjoint is a Quillen adjunction if the following equivalent conditions are satisfied:

- L preserves cofibrations and trivial cofibrations,
- R preserves fibrations and trivial fibrations,
- L preserves cofibrations and R preserves fibrations,
- L preserves trivial cofibrations and R preserves trivial fibrations.

If (L, R) is a Quillen adjunction, we sometimes say that L is a left Quillen functor and R is a right Quillen functor. This can be useful if we know that there exists a Quillen adjunction, but we are only interested in one of the two functors of the adjunction.

Remark. For the sake of completeness, we will provide a full proof that the conditions in the definition above are indeed equivalent. This result is usually left as an exercise in most literature on the subject, or only proven partially.

Proposition 4.2. The conditions in the definition above are equivalent.

Proof. We first show that a left adjoint $L : \mathcal{M}_2 \to \mathcal{M}_1$ preserves trivial cofibrations if and only if its right adjoint $R : \mathcal{M}_1 \to \mathcal{M}_2$ preserves fibrations.

Suppose that L preserves trivial cofibrations and let $f : X \to Y$ be a fibration in $\mathcal{M}_1$ and $g : \tilde{X} \to \tilde{Y}$ a trivial cofibration in $\mathcal{M}_2$. Since (L, R) is a pair of adjoint functors, we have a natural isomorphism $\hom_{\mathcal{M}_2}(LA, B) \simeq \hom_{\mathcal{M}_1}(A, RB)$ for $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_2$. This means that we have the left diagram if and only if we have the right diagram of the following two diagrams.

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "g"'] & RX \arrow[d, "Rf"] \\ \tilde{Y} \arrow[r] & RY \end{tikzcd} \qquad \begin{tikzcd} L\tilde{X} \arrow[r] \arrow[d, "Lg"'] & X \arrow[d, "f"] \\ L\tilde{Y} \arrow[r] & Y \end{tikzcd}$$

Since f is a fibration and g is a trivial cofibration, the right diagram above has a lift $h : L\tilde{Y} \to X$ because fibrations have the right lifting property with respect to trivial cofibrations. Since we have a isomorphism $\hom_{\mathcal{M}_1}(L\tilde{Y}, X) \simeq \hom_{\mathcal{M}_1}(\tilde{Y}, RX)$, there exists a map $\tilde{h} : Y \to RX$ and by naturality of the isomorphism between the hom sets this is a lift of the left diagram. Therefore Rf has the right lifting property with respect to the trivial cofibration g. By 3.10 we know that fibrations are given exactly by maps that have the right lifting property with respect to trivial cofibrations. Therefore Rf is a fibration and therefore R preserves fibrations.

Suppose now that R preserves fibrations, that is to say that Rf is a fibration. We want to show that Lg is a trivial cofibration. We consider the same two commutative diagrams as above.

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "g"'] & RX \arrow[d, "Rf"] \\ \tilde{Y} \arrow[r] & RY \end{tikzcd} \qquad \begin{tikzcd} L\tilde{X} \arrow[r] \arrow[d, "Lg"'] & X \arrow[d, "f"] \\ L\tilde{Y} \arrow[r] & Y \end{tikzcd}$$

The left diagram has a lift $h : \tilde{Y} \to RX$ because g is a trivial cofibration and therefore has the left lifting property with respect to the fibration Rf. As before, this yields a lift $\tilde{h} : L\tilde{Y} \to X$ for the right diagram. Therefore Lg has the left lifting property with respect to all fibrations, since f is an arbitrarily chosen fibration of $\mathcal{M}_1$. By 3.10 this means that Lg is a trivial cofibration. Thus L preserves trivial cofibrations as a direct result of R preserving fibrations.

So far we have shown that L preserves trivial cofibrations precisely if R preserves fibrations. We claim that L preserves cofibrations if and only if R preserves trivial fibrations. This can be shown to be true in an identical manner as above, using the same diagrams as above.

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "g"'] & RX \arrow[d, "Rf"] \\ \tilde{Y} \arrow[r] & RY \end{tikzcd} \qquad \begin{tikzcd} L\tilde{X} \arrow[r] \arrow[d, "Lg"'] & X \arrow[d, "f"] \\ L\tilde{Y} \arrow[r] & Y \end{tikzcd}$$

However, we now require f to be a trivial fibration and g a cofibration.

Suppose L preserves cofibrations. This means that Lg is a cofibration. Since cofibrations have the left lifting property with respect to trivial fibrations, see 3.10, there exists a lift $h : L\tilde{Y} \to X$ such that the right diagram above is commutative. By virtue of the natural isomorphism $\hom_{\mathcal{M}_1}(L\tilde{Y}, X) \simeq \hom_{\mathcal{M}_1}(\tilde{Y}, RX)$, there exists a lift $\tilde{h} : \tilde{Y} \to RX$ such that the left diagram above is commutative. Thus, Rf has the right lifting property with respect to g and since g is an arbitrarily chosen cofibration in $\mathcal{M}_2$ we conclude that Rf has the right lifting property with respect to all cofibrations in $\mathcal{M}_2$. By 3.10 we conclude that Rf is a trivial fibration. Therefore L preserving cofibrations implies that R preserves trivial fibrations.

Conversely, suppose R preserves trivial fibrations. This implies that Rf is a trivial fibration. Since g is a cofibration and cofibrations have the left lifting property with respect to trivial fibrations, there exists a lift for the left diagram above. Once again, using the natural isomorphism between hom sets coming from the fact that L and R are adjoint functors, we find a lift for the right diagram. This means that Lg has the left lifting property with respect to the trivial fibration f. Since f is an arbitrarily chosen trivial fibration, we conclude that Lg has the left lifting property with respect to all trivial fibrations and by Theorem 3.10 we conclude that Lg is a cofibration. Therefore L preserves cofibrations.

What we have shown so far is that

- L preserves trivial cofibrations if and only if R preserves fibrations,
- L preserves cofibrations if and only if R preserves trivial fibrations.

To show that the four conditions in Definition 4.1 are equivalent, we proceed as follows.

Suppose that L preserves cofibrations and trivial cofibrations. Using the two claims that we have shown so far, we immediately find that R preserves fibrations and trivial fibrations.

Suppose R preserves cofibrations and trivial fibrations. Using the first of the two claims we have shown earlier we conclude that L preserves cofibrations as a result of R preserving trivial fibrations.

Suppose L preserves cofibrations and R preserves fibrations. Using the both claims that we have shown, we find that L preserves trivial cofibrations and R preserves trivial fibrations.

Suppose L preserves trivial cofibrations and R preserves trivial fibrations. Using the second claim we have proven, we arrive back at the first condition from Definition 4.1, namely that L preserves cofibrations and acyclic cofibrations. ■

Definition 4.3. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two model categories. Given a left Quillen functor $F : \mathcal{M}_1 \to M_2$ and a right Quillen functor $G : \mathcal{M}_2 \to \mathcal{M}_1$ we define the total left derived functor $LF : \mathrm{Ho}(\mathcal{M}_1) \to \mathrm{Ho}(\mathcal{M}_2)$ as the composition

$$\mathrm{Ho}(\mathcal{M}_1) \overset{\mathrm{Ho}(Q)}{\to} \mathrm{Ho}((\mathcal{M}_1)_c) \overset{\mathrm{Ho}(F)}{\to} \mathrm{Ho}(\mathcal{M}_2),$$

where Q is the cofibrant replacement functor of $\mathcal{M}_1$, and the total right derived functor $RG : \mathrm{Ho}(\mathcal{M}_2) \to \mathrm{Ho}(\mathcal{M}_1)$ as the composition

$$\mathrm{Ho}(\mathcal{M}_2) \overset{\mathrm{Ho}(R)}{\to} \mathrm{Ho}((\mathcal{M}_2)_f) \overset{\mathrm{Ho}(G)}{\to} \mathrm{Ho}(\mathcal{M}_1),$$

where R is the fibrant replacement functor of $\mathcal{M}_2$.

Definition 4.4. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two model categories and let

$$(F, G) : \mathcal{M}_1 \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{M}_2$$

be a Quillen adjunction with F left adjoint and G right adjoint.

The Quillen adjunction (F, G) is a Quillen equivalence if the following equivalent conditions are satisfied.

- The total left derived functor $LF : \mathrm{Ho}\mathcal{M}_1 \to \mathrm{Ho}\mathcal{M}_2$ is an equivalence of the homotopy categories,
- The total right derived functor $RG : \mathrm{Ho}\mathcal{M}_2 \to \mathrm{Ho}\mathcal{M}_1$ is an equivalence of the homotopy categories,
- Given a cofibrant object $C \in \mathcal{M}_1$ and a fibrant object $D \in \mathcal{M}_2$, a morphism $C \to G(D)$ is a weak equivalence in $\mathcal{M}_1$ precisely when the adjunct morphism $F(C) \to D$ is a weak equivalence in $\mathcal{M}_2$,

- The following two conditions hold:
  1. The derived adjunction unit is a weak equivalence, in that for every cofibrant object $C \in \mathcal{M}_1$, the composite $C \xrightarrow{\eta_C} G(F(C)) \to G(F(C)^{fib})$ (of the component at C of the adjunction unit η with a fibrant replacement $R(F(C) \xrightarrow{\simeq} F(C)^{fib})$) is a weak equivalence in $\mathcal{M}_1$,
  2. The derived adjunction counit is a weak equivalence, in that for every fibrant object $D \in \mathcal{M}_2$, the composite $G(F(D)^{cof}) \to F(G(D)) \xrightarrow{\epsilon_D} D$ (of the component at D of the adjunction counit ϵ with cofibrant replacement $F(G(D)^{cof} \xrightarrow{\simeq} G(D))$) is a weak equivalence in $\mathcal{M}_2$.

Definition 4.5. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two model categories. We say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are Quillen equivalent if there exists a zig-zag of Quillen equivalences between them.

Note that not every equivalence between homotopy categories of model categories lifts to a Quillen equivalence on the level of the model categories. We will present one example of two model categories that have equivalent homotopy categories but are not Quillen equivalent.

§4.2 Equivalence of model categories

As promised, we will now discuss how exactly the notion of Quillen equivalence is the "correct" notion of equivalence for model categories. This part of the thesis is by far the most informal and this is for a good reason. A rigorous treatment of some of the concepts we will touch on is not in line with the main intention of this thesis, which is to provide a gentle introduction to model categories and present an interesting example of model categories that are not Quillen equivalent and fail to be so in a non-trivial way. Nonetheless, clarifying the question of finding the right notion of equivalence for model categories is an important and interesting issue that deserves some attention and we will make an attempt at pointing towards a handful of useful and detailed works on ∞-categories, hammock localization and other concepts that will be tangentially mentioned in this part of the thesis.

If we remind ourselves of the definition of a model category given before, it is hard not to realize that a categorical equivalence between two model categories is in no way guaranteed to preserve the model structure of the two model categories. In fact, it suffices to recall that a category can be endowed with different model structures that have completely different distinguished classes of weak equivalences, fibrations and cofibrations.

Consider the category of topological spaces. We have briefly touched on two model structures for Top in 3.29 and 3.30. Finding a weak equivalence in the model structure that yields the homotopy category of CW-complexes that is not a weak equivalence in the model structure that yields the homotopy category of topological spaces, that is to say finding a weak homotopy equivalence in the usual sense that is not a homotopy equivalence in the usual sense, is far from difficult.

Example 4.6. We consider two topological spaces $X = \mathbb{N}$ and $Y = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with the subspace topology coming from $\mathbb{R}$, endowed with the standard topology. We consider the map $f : X \to Y$ that maps 0 to 0 and $x \in X$ to $\frac{1}{x} \in Y$. This map is continuous:

Let $U \neq \emptyset$ be an open subset of Y. Remark that $U \neq \{0\}$ since any open subset of $\mathbb{R}$ containing 0 will contain the set $\{\frac{1}{N}, \frac{1}{N+1}, \ldots\}$ for some $N \in \mathbb{N}$ large enough. Clearly we have $f^{-1}(U) = \{n \in \mathbb{N} : f(n) \in U\}$ and this is an open subset of X. This follows from

$f^{-1}(U) = X \cap \bigcup_{\substack{n\in\mathbb{N} \\ f(n)\in U}} (n - \frac{1}{4}, n + \frac{1}{4})$ since $\bigcup_{\substack{n\in\mathbb{N} \\ f(n)\in U}} (n - \frac{1}{4}, n + \frac{1}{4})$ is an open subset of $\mathbb{R}$. Thus, f is a morphism in the category of topological spaces.

For f to be a homotopy equivalence, there must exist some map $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ homotopic to id_X. Let $H : I \times X \to X$ be the homotopy from $g \circ f$ to id_X, with $H(0, x) = g \circ f(x)$ and $H(1, x) = x$. We have $H(0, 0) = g(0)$ and $H(1, 0) = 0$. Since $\{0\}$ is a path component of X and $H(t, 0)$ defines a path from 0 to $g(0)$, we conclude that $g(0) = 0$.

Thus, if f is a homotopy equivalence, then there exists a continuous map $g : Y \to X$ such that $g(0) = 0$. Now, let us suppose that there exists $y \in Y \setminus \{0\}$ such that $g(y) = 0$. If f is a homotopy equivalence, then so is g and therefore induces a bijection between the set of path components of Y and X. Since g sends 0 to 0 and y to 0, this implies that y and 0 are in the same path component of Y, which is not possible, since the path components of Y are all given by singletons. This means that $g^{-1}(\{0\}) = \{0\}$.

By assumption, g is continuous and thus $g^{-1}(\{0\}) = \{0\}$ is an open subset of Y. But this is clearly not true. Any open set of Y is given by the intersection of an open subset of $\mathbb{R}$ with Y. Any open set U of $\mathbb{R}$ containing 0 must contain $(-\epsilon, +\epsilon)$ for some $\epsilon > 0$. But this implies that for N large enough, $\frac{1}{N} \in Y \cap (-\epsilon, +\epsilon) \subset Y \cap U \neq \{0\}$ and thus, $\{0\}$ is not an open set of Y.

On the other hand, f is trivially a weak homotopy equivalence, which we leave as an exercise to the reader to verify.

Despite the above example being rather trivial, it neatly illustrates the importance of a Quillen equivalence when talking about equivalent model categories. In fact, it is thanks to the notion of Quillen equivalence that we are able to "categorify" model categories in the form of the double category of model categories.

Definition 4.7. We define the double category of model categories as consisting of model categories as objects, left Quillen functors as vertical morphisms, right Quillen functors as horizontal morphisms and natural transformations between compositions of functors as 2-morphisms.

Remark. In case the reader is unfamiliar with the notion of double categories, we refer to the appendix, in particular Definition 6.13.

There are more places where Quillen equivalences show up in a deep and substantial way. In particular, every Quillen equivalence between two model categories can be turned into an equivalence of $(\infty, 1)$-categories.

Definition 4.8. An $(\infty, 1)$-category is a simplicial set $S_\bullet$, such that for $0 < i < n$, every map of simplicial sets $f : \Lambda^n_i \to S_\bullet$ can be extended to a map $\tilde{f} : \Delta^n \to S_\bullet$, where Δ^n is the standard n-simplex and Λ^n_i is the i-th horn.

Remark. In case the reader is unfamiliar with simplicial sets, we refer to the appendix, in particular Definition 6.18.

Suppose we are given a model category $\mathcal{M}$ with weak equivalences $\mathcal{W}$. We can construct an $(\infty, 1)$-category $\mathcal{M}_\infty$ along with a map $\mathcal{M} \to \mathcal{M}_\infty$, characterized by the universal property that given any ∞-category C there exists a natural fully faithful map $\mathrm{Func}(\mathcal{M}_\infty, C) \to \mathrm{Func}(\mathcal{M}, C)$ for which the essential image comes from functors $\mathcal{M} \to C$ that send weak equivalences to equivalences. A detailed treatment with full proofs can be found in [12].

This construction starts out by considering the so-called hammock localization of the model category $\mathcal{M}$ with respect to $\mathcal{W}$, which yields a simplicial category. For details on the hammock localization we refer to [10]. Bergner [3] showed that there is a model category structure for the category of simplicial categories, which is cofibrantly generated. We consider a fibrant replacement (with respect to the Bergner model structure) of our simplicial category. By looking at the full simplicial subcategory, which has fibrant cofibrant objects, we get an ∞-category by taking the homotopy coherent nerve of this simplicial subcategory. This is the ∞-category underlying the model category $\mathcal{M}$. For a more in-depth overview of localization of model categories, the reader may want to consult [12] or [15].

§5 Model categories that are not Quillen equivalent

We are now turning our attention to two rings. Namely $\mathbb{Z}/p^2$ and $(\mathbb{Z}/p)[\epsilon]/(\epsilon^2)$. We are interested in the stable module categories $\mathrm{Stmod}(\mathbb{Z}/p^2)$ and $\mathrm{Stmod}((\mathbb{Z}/p)[\epsilon]/(\epsilon^2))$, which turn out not to be Quillen equivalent. While there are many examples of model categories that are not Quillen equivalent, there are not many examples that are known to be not Quillen equivalent in a non-trivial manner, as mentioned in [1].

From here on out, we fix a prime p and fix two rings $R := \mathbb{Z}/p^2$ and $R_\epsilon := (\mathbb{Z}/p)[\epsilon]/(\epsilon^2)$. Additionally, we denote by $\mathcal{M}$ the stable module category of R and by $\mathcal{M}_\epsilon$ the stable module category of R_ϵ. These are pointed model categories and we have already introduced them in 3.41. It is easy to see that the rings R and R_ϵ are Frobenius rings.

The homotopical equivalence of the model categories $\mathcal{M}$ and $\mathcal{M}_\epsilon$ has been known for quite some time and is reasonably obvious. Schlichting [18] showed that the model categories themselves are not Quillen equivalent, using K-theory. Instead of relying on the computationally heavy machinery from K-theory, we will follow the approach taken by Dugger & Shipley [7]. In their paper they provide a more elegant argument to show that $\mathcal{M}$ and $\mathcal{M}_\epsilon$ are not Quillen equivalent. We will first fill in some missing details in the proof that the homotopy categories associated to our two model categories are indeed equivalent. Following this, the rest of this thesis focuses on giving an overview of their approach and present their main results.

As we have previously discussed in Theorem 3.41, there exists a model structure on the category of R-modules and R_ϵ-modules whose cofibrations are the injections, fibrations are the surjections and weak equivalences are the so called stable homotopy equivalences.

The two model categories $\mathcal{M}$ and $\mathcal{M}_\epsilon$ are stable model categories, which is to say that the suspension and loop functor on the associated homotopy categories $\mathrm{Ho}(\mathcal{M})$ and $\mathrm{Ho}(\mathcal{M}_\epsilon)$ are self-equivalences. Seeing as we have not even mentioned the existence of these functors on the homotopy category of a model category, we will provide an overview of where these suspension functors come from in the general case of pointed model categories.

To construct the suspension functors for the homotopy category of any pointed model category $\mathcal{M}$, the fact that the homotopy category of a pointed model category can be given a closed $\mathrm{Ho}(\mathsf{SSet}_*)$-module structure, where SSet_* is the category of pointed simplicial sets, is of crucial importance. It allows defining suspension functors for $\mathrm{Ho}(\mathcal{M})$ as a closed action of $\mathrm{Ho}(\mathsf{SSet}_*)$ on $\mathrm{Ho}(\mathcal{M})$.

Since we chose not to give a detailed presentation of the theory of simplicial sets, apart from providing the most basic definitions in the appendix, and due to the technical nature of establishing that $\mathrm{Ho}(\mathcal{M})$ can indeed be given the structure of a $\mathrm{Ho}(\mathsf{SSet}_*)$-module, we will not show the construction of the loop and suspension functors for the homotopy category of any pointed model category. The existence of suspension functors for the homotopy category of any pointed model category is however of little concern to us. The two model categories that are of interest to us, allow for an explicit construction of the suspension and loop functor without appealing to the general existence result.

For an extensive treatment of the $\mathrm{Ho}(\mathsf{SSet}_*)$-module structure of $\mathrm{Ho}(\mathcal{M})$ for any pointed model category $\mathcal{M}$ and the existence of suspension functors for $\mathrm{Ho}(\mathcal{M})$ the reader may want to consult [13], Chapter 5 & 6.

§5.1 Homotopy category equivalence

Before discussing how $\mathcal{M}$ and $\mathcal{M}_\epsilon$ are not Quillen equivalent we will show that $\mathrm{Ho}(\mathcal{M})$ and $\mathrm{Ho}(\mathcal{M}_\epsilon)$ are equivalent. Of course if those two homotopy categories weren't equivalent, by definition 4.4 they wouldn't be Quillen equivalent and the example would not be of much interest.

Suppose M is a module over $\mathbb{Z}/p^2$. Let ΓM denote $(\mathrm{Ann}_M p)/pM$, where $\mathrm{Ann}_M p$ are all the elements of M that are annihilated by multiplication by p. ΓM is naturally a $\mathbb{Z}/p$-vector space, since any element of $\mathrm{Ann}_M p$ is of order p in M and by taking the quotient of this annihilator with pM we make sure that for $x, y \in \mathbb{Z}/p^2$ such that $\varphi(x) = \varphi(y)$, where $\varphi : \mathbb{Z}/p^2 \to \mathbb{Z}/p$ is the operation mod p, it holds that for any $m \in \mathrm{Ann}_M p$ we have $xm - ym = (x - y)m = (kp)m \in pM$ and thus $\varphi(x)m = \varphi(y)m$ in ΓM and thus the scalar multiplication $\cdot : \mathbb{Z}/p \times M \to M$ induced by $\cdot : \mathbb{Z}/p^2 \times M \to M$ is well defined.

For any M, let $C_*(M)$ denote the chain complex graded over $\mathbb{Z}$ that has M in every dimension and the differentials are given by multiplication by p. This means that ΓM is the homology of $C_*(M)$ in dimension 0.

Since $C_*(M)$ consists of M in every dimension, looking at the homology at any dimension looks as follows:

$$\ldots \to \ker(M \xrightarrow{\cdot p} M)/(\mathrm{im}(M \xrightarrow{\cdot p} M)) \to \ldots.$$

Since $\ker(M \xrightarrow{\cdot p} M) = \{x \in M : px = 0\}$ and $\mathrm{im}(M \xrightarrow{\cdot p} M) = pM$, we have indeed that the homology is equal to ΓM.

Lemma 5.1. Let M be a module over $\mathbb{Z}/p^2$. M is isomorphic to $\Gamma M \oplus F$, where F is a free module.

Proof. Let $\{v_i\}$ be a $\mathbb{Z}/p$ basis for pM, which is justified by virtue of pM being a $\mathbb{Z}/p$-module.For any i there exists some $w_i \in M$ for which $pw_i = v_i$. Let F be generated by all w_i, which form a free basis for F. This is true because suppose that $\sum_{i=0}^{n} r_i w_i = 0$, then this implies that $\sum_{i=0}^{n} pr_i w_i = \sum_{i=0}^{n} r_i v_i = 0$, which implies that $r_i = 0$ for all i, since v_i form a $\mathbb{Z}/p$ basis for pM.

The canonical inclusion $\mathrm{Ann}_M p \to M$ induces the map $\Gamma M \to M/F$. We want to show that this is an isomorphism. Consider the short exact sequence of chain complexes

$$0 \to C_*(F) \to C_*(M) \to C_*(M/F) \to 0,$$

where $C_*(F)$ is exact since F is free. Using the zig-zag lemma we find that $\Gamma M = \Gamma(M/F)$. But $\Gamma(M/F) = M/F$ since multiplying by p restricts to the zero map for M/F.

M/F is a $\mathbb{Z}/p$-vector space, which means we can choose a basis $\{a_j\}$. For all j there exists some $b_j \in M$ such that $\pi(b_j) = a_j$ and for which we have $pb_j = 0$, where π is the quotient map $M \to M/F$. From here we have a splitting for the short exact sequence $0 \to F \to M \to M/F \to 0$ by mapping a_j to b_j. If we can show that the map $a_j \mapsto b_j$ is a $\mathbb{Z}/p^2$-module homomorphism we are done, since it is obvious that $\pi \circ (a_j \mapsto b_j) = \mathrm{id}_{M/F}$ and therefore we have $M \simeq F \oplus M/F$.

Since $\varphi : \mathbb{Z}/p^2 \to \mathbb{Z}/p$ with $\varphi(x) = x \bmod p$ is a ring homomorphism, any $\mathbb{Z}/p$-module structure can be pulled back to a $\mathbb{Z}/p^2$-module structure. Thus both M/F and pM, which are $\mathbb{Z}/p$-modules are naturally $\mathbb{Z}/p^2$-modules.

This means that any $\mathbb{Z}/p$-module morphism $f : M/F \to pM$ can be given the structure of a $\mathbb{Z}/p^2$-module morphism $\tilde{f} : M/F \to pM$, where $\tilde{f}(r \cdot x) := f(\varphi(r) \cdot x)$ for any $r \in \mathbb{Z}/p^2$ and $x \in M/F$.

It is obvious that extending the map $a_j \mapsto b_j$ by linearity to a map $f : M/F \to M$ doesn't change the fact that $pf(x) = 0$, since any $x \in M/F$ is of the form $\sum_{i=0}^{n} r_i a_i$ and thus $pf(x) = p \sum_{i=0}^{n} r_i p b_{j(i)} = 0$ and therefore f is in fact a map from $M/F \to pM$, since pM contains all elements of order p of M. ∎

Our next short term aim is to show that $\mathrm{Vect}_{\mathbb{Z}/p}(V,W) \to \mathrm{Ho}(\mathcal{M})(V,W)$ is an isomorphism for any two vector spaces V and W over $\mathbb{Z}/p$. Before we show this we need to establish some basic facts about the model category we are working with.

Lemma 5.2. Let M be an R-module. Then there exists a very good path object for M.

Proof. Let M be an R-module and let $F \xrightarrow{f} M$ be any surjection from a free module F onto M. Let PM denote the direct sum $M \oplus F$. We have a natural inclusion $i : M \to PM$.

Let $\pi : PM \to M \oplus M$ be the map given by $\pi(m,f) = \nabla_M(m) + if$, where ∇_M is the diagonal on M.

This means that $\pi i = \nabla_M$ and therefore πi is a factorization of the diagonal ∇_M.

Additionally $M \to PM$ is a trivial cofibration, since cofibrations are exactly injective maps and triviality follows from the fact that the inclusion $M \to M \oplus P$ is a stable equivalence for any projective module P and since F is a free module it is projective, meaning that $M \to M \oplus F$ is a stable equivalence.

Finally, $PM \to M \oplus M$ is a fibration since it is surjective. In fact, suppose we choose any $(m_1, m_2) \in M \oplus M$. Since $m_2 - m_1 \in M$ and $F \to M$ is surjective, there exists $m \in F$ such that $f(m) = m_2 - m_1$. It is easy to see that $\pi(m_1, m) = (m_1, m_1) + (0, f(m)) = (m_1, m_1 + m_2 - m_1) = (m_1, m_2)$ and thus $PM \to M \oplus M$ is surjective, which means that by virtue of fibrations being exactly surjective maps this is in fact a fibration.

Thus, PM is a very good path object for M. ∎

In the following result, the notion of a coequalizer of two maps appears. The notation will be explained and introduced step by step in the proof of the lemma.

Lemma 5.3. Let M, PM and F be as above along with the maps that we just introduced. Then for any R-module J

$$\mathrm{coeq}(\mathcal{M}(J, PM) \rightrightarrows \mathcal{M}(J, M)) \to \mathrm{Ho}(\mathcal{M})(J, M)$$

is an isomorphism.

Proof. First we comment on the notion of a coequalizer. Given two parallel morphisms between two objects $f, g : X \to Y$, the coequalizer is an object Q together with a morphism $q : Y \to Q$ such that $q \circ f = q \circ g$ and such that the pair Q, q is universal in the sense that given any other pair (Q', q') there exists a unique morphism $h : Q \to Q'$ such that $u \circ q = q'$. This is represented in the following diagram:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"] & Q \arrow[d, "u"] \\ & & Q' \end{tikzcd}$$

This means that in our case, we are interested in an object Q and a morphism $q : \mathrm{Hom}_{\mathcal{M}}(J, M) \to Q$ such that $q \circ p_0 = q \circ p_1$ and such that (Q, q) is universal as defined above.

Since $\mathcal{M}$ is the category of R-modules and R is commutative, $\mathrm{Hom}_{\mathcal{M}}(X, Y)$ is an R-module for any R-modules X and Y. Thus, the coequalizer of two parallel R-module homomorphisms f, g is given by the cokernel of $f - g$.

We recall that for the cofibrantly generated model structure on the category of R-modules, two maps are stably equivalent if and only if they are left or right homotopic. This means that their difference factors through a projective module if and only if there exists a left or right homotopy. Since all objects in this model category are both fibrant and cofibrant, by 3.23 the existence of a left homotopy is equivalent to the existence of a right homotopy.

For any R-module J, the two parallel morphisms $\mathrm{Hom}_{\mathcal{M}}(J, PM) \to \mathrm{Hom}_{\mathcal{M}}(J, M)$ are given by $\tilde{p}_0$ and $\tilde{p}_1$, where $\tilde{p}_0$ is left composition with $p_0 : PM \to M$, defined by $p_0(m, f) = m$ and $\tilde{p}_1$ is left composition with $p_1 : PM \to M$, defined by $p_1(m, f) = \alpha(f)$, where $\alpha : F \to M$ is the surjection from F to M.

Thus, for any map $f \in \mathrm{Hom}_{\mathcal{M}}(J, PM)$ we can construct two maps $\tilde{p}_0(f)$ and $\tilde{p}_1(f)$ which are in $\mathrm{Hom}_{\mathcal{M}}(J, M)$.

The coequalizer of $\tilde{p}_0$ and $\tilde{p}_1$ is given by the cokernel of $\tilde{p}_0 - \tilde{p}_1$, which is the quotient module $\mathrm{Hom}_{\mathcal{M}}(J, M)/\mathrm{Im}(\tilde{p}_0 - \tilde{p}_1)$.

Suppose now that we have two morphisms $f, g \in \mathrm{Hom}_M(J, M)$, such that $f - g \in \mathrm{Im}(\tilde{p}_0 - \tilde{p}_1)$, i.e. f and g are two representatives of the same equivalence class of $\mathrm{coker}(\tilde{p}_0 - \tilde{p}_1)$. This implies that there exists a map $h : J \to PM$ such that $(\tilde{p}_0 - \tilde{p}_1)(h) = f - g$ and thus $f - g$ factors through the projective PM. This implies that f and g are stably equivalent and thus by Theorem 3.41 there exists a homotopy between f and g and thus, f and g are mapped to the same equivalence class in $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(J, M)$.

Likewise, suppose we have two representatives $f, g : J \to M$ of the same equivalence class of $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(J, M)$. This implies that there exists a homotopy between f and g. By Lemma 3.21 we can pick any good path object to exhibit the homotopy between f and g, therefore we can choose PM as our good path object.

We have a homotopy $\eta : f \Rightarrow g$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & J \arrow[dl, "f"'] \arrow[d] \arrow[dr, "g"] & \\
M & PM \arrow[l, "\tilde{p}_0"] \arrow[r, "\tilde{p}_1"'] & M
\end{tikzcd}$$

Therefore, we conclude that $f - g = \tilde{p}_0(\eta) - \tilde{p}_1(\eta)$ and thus $f - g \in \mathrm{Im}(\tilde{p}_0 - \tilde{p}_1)$ and thus f and g belong to the same class of $\mathrm{coker}(\tilde{p}_0 - \tilde{p}_1)$.

Thus, both directions of the map $\mathrm{coker}(\tilde{p}_0 - \tilde{p}_1) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(J, M)$ are well-defined.

The fact that this map is a bijection follows immediately from the fact that any map $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(J, M)$ factors through PM and thus must be in the cokernel of $\tilde{p}_0 - \tilde{p}_1$. ∎

Theorem 5.4. Let V and W be two vector spaces over $\mathbb{Z}/p$. Then $\mathrm{Vect}(V, W) \to \mathrm{Ho}(\mathcal{M})(V, W)$ is an isomorphism.

Proof. We claim that the two parallel morphisms $\mathcal{M}(V, PW) \rightrightarrows \mathcal{M}(V, W)$ are the same. This is indeed true, since the only way for a morphism between two vector spaces $V \to W$ over $\mathbb{Z}/p$ to factor through a free module over $\mathbb{Z}/p^2$ is the zero map, which means that the two parallel morphisms are the same and more precisely they are given by $PW := W \oplus F \ni (w, f) \mapsto w \in W$.

We will justify why a linear map between $\mathbb{Z}/p$-vector spaces that factors through a free module over $\mathbb{Z}/p^2$ must be the zero map. Let $f : V \to W$ be a linear map with V

and W $\mathbb{Z}/p$-vector spaces, with $f = g \circ h$ factoring through some free $\mathbb{Z}/p^2$-module M, where $g : M \to W$ and $h : V \to M$.

From here on, we regard V, W and M as $\mathbb{Z}/p^2$-modules and the linear map is trivially a $\mathbb{Z}/p^2$-module homomorphism. The $\mathbb{Z}/p^2$-module structure comes from the fact that we have a ring homomorphism $\varphi : \mathbb{Z}/p^2 \to \mathbb{Z}/p$, given by modulus p, and $\mathbb{Z}/p^2$ acts on V and W as follows: For any $r \in \mathbb{Z}/p^2$ and x an element of V or W, we take $r \cdot x = \varphi(r) \cdot x$, where the right-hand side is given by the $\mathbb{Z}/p$-vector space structure of V or W.

Suppose that there exists $x \in V$ such that $f(x) \neq 0$. Since $px = 0$, we know that $pg(x) = 0$, which implies that $h(x)$ is of order p in M. Since M is a free module, there exists a basis for M. Thus we know that there exists some $y \in M$ such that $py = h(x)$. Therefore we have $0 \neq f(x) = gh(x) = g(py) = pg(y) = 0$, since any non-zero element of W is of order p. Thus, f has to be the zero map.

Therefore, $\mathrm{coeq}(\mathcal{M}(V, PW) \overset{\to}{\to} \mathcal{M}(V, W)) = \mathrm{Vect}_{\mathbb{Z}/p}(V, W)$ and thus by the previous proposition we have $\mathrm{Vect}_{\mathbb{Z}/p}(V, W) \simeq \mathrm{Ho}(\mathcal{M})(V, W)$. ∎

We have shown that we have $\mathrm{Vect}_{\mathbb{Z}/p}(V, W) = \mathrm{Ho}(\mathcal{M})(V, W)$. If we show that $\mathrm{Vect}_{\mathbb{Z}/p}(V, W) = \mathrm{Ho}(\mathcal{M}_\epsilon)(V, W)$ then we have effectively shown that $\mathrm{Ho}(\mathcal{M}) \simeq \mathrm{Ho}(\mathcal{M}_\epsilon)$. An observant reader may point out that we have focused all of our efforts on how morphisms are mapped between $\mathrm{Vect}_{\mathbb{Z}/p}$ and $\mathrm{Ho}(\mathcal{M})(V, W)$. This is of little importance, since the appropriate map of objects has already been introduced earlier, specifically the map Γ that assigns to each module M the module ΓM.

Proving that $\mathrm{Vect}(V, W) = \mathrm{Ho}(\mathcal{M}_\epsilon)(V, W)$ is easier than doing so for $\mathcal{M}$. Every module over R_ϵ is naturally a $\mathbb{Z}/p$-vector space, which simplifies the arguments significantly. Instead of defining ΓM as $(\mathrm{Ann}_M p)/pM$, we define ΓM as $(\mathrm{Ann}_M \epsilon)/\epsilon M$ and the rest follows from the exact same reasoning as above for the ring R.

Therefore we have

Theorem 5.5. $\mathrm{Ho}(\mathcal{M}) \simeq \mathrm{Ho}(\mathcal{M}_\epsilon)$.

§5.2 Not Quillen-equivalent

Having shown that the two model categories $\mathcal{M}$ and $\mathcal{M}_\epsilon$ have equivalent homotopy categories, we will now give an overview of why the model categories themselves are not Quillen equivalent. This requires a lot of machinery, much of which Brooke & Shipley do not cover in their paper and much of which has been introduced by earlier papers by the same authors. We will refer to these earlier papers whenever convenient and helpful to the reader.

The approach that Brooke & Shipley take to show that the model categories $\mathcal{M}$ and $\mathcal{M}_\epsilon$ are not Quillen equivalent consists of multiple steps. We will present each step as its own subsection.

From here on out, T denotes a generic Frobenius ring and we denote by $[M, N]$ the stable homotopy classes of maps, i.e. $[M, N] = \mathrm{Hom}_{\mathrm{Stmod(T)}}(M, N)/\sim$, where $f \sim g$ if and only if there exists a left or right homotopy between the two, or equivalently if $f - g$ factors through a projective, that is to say that f and g are stably equivalent.

5.2.1 Quillen equivalence with model category of dg-modules

The first step consists of showing that the model category of stable modules over a Frobenius ring that comes with a compact weak generator is Quillen equivalent to a certain model category of dg-modules over a dga.

Before we show this result, we require a few more definitions.

Definition 5.6. Let M be a module in $\mathrm{Stmod}(T)$. We say that M is *compact* if the map $\bigoplus_{i\in I}[M,N_i] \to [M,\bigoplus_{i\in I} N_i]$ is an isomorphism for any collection of objects $\{N_i\}_{i\in I}$.

Definition 5.7. Let M be a T-module. We denote by $I(M)$ the object coming from the functorial factorization of $M \to *$ into $M \to I(M) \to 0$, where the first map is a cofibration and the second map is a trivial fibration.

Analogously, we denote by $P(M)$ the object coming from the functorial factorization of $* \to M$ into $* \to P(M) \to M$, where the first map is a trivial cofibration and the second a fibration.

We define $\Sigma M := \mathrm{coker}(M \to I(M))$ and $\Omega M := \ker(P(M) \to M)$.

Remark. An observant reader may notice that in the definition above, we are essentially repeating the argument that we used to show the existence of cofibrant and fibrant replacements in the proof of Proposition 3.9. The reason for not referring to $I(M)$ as the cofibrant replacement and $P(M)$ as the fibrant replacement has to do with how the cofibrant and fibrant replacements are defined. In the statement of Proposition 3.9 we consider some object X of a model category $\mathcal{M}$ and introduce a fibrant object RX along with a cofibrant object QX, both of which are weakly equivalent to X. In the proof of the proposition we use the two functorial factorizations of the model category $\mathcal{M}$ to produce two maps $\emptyset \to QX \to X$ and $X \to RX \to *$, with the first map given as a composition of a cofibration followed by a trivial fibration and the second as a composition of a trivial cofibration followed by a fibration.

We chose not to require the weak equivalence $QX \to X$ to be a fibration and the weak equivalence $X \to RX$ to be a cofibration in our definition of the cofibrant and fibrant replacement to stick to the standard treatment of the fibrant and cofibrant replacement in most literature and references on model categories.

Therefore, to avoid any potential confusion, in the definition above we decided to appeal to the functorial factorization instead of the fibrant and cofibrant replacements. This guarantees a lack of doubts over why the weak equivalences between $I(M)$, M and $P(M)$ are in fact trivial (co)fibrations and not merely weak equivalences.

Remark. At this point, we deem it sensible to point out that in the definition above, Ω and Σ are the loop and suspension functors of the homotopy category of the model category $\mathrm{Stmod}(T)$. Earlier, we briefly touched on the existence of these functors for any pointed model category as a result of homotopy categories of pointed model categories having a $\mathrm{Ho}(\mathrm{SSet}_*)$-module structure and remarked that the context of the model categories $\mathcal{M}$ and $\mathcal{M}_\epsilon$ allows identifying these functors explicitly, by which we mean precisely Ω and Σ as given in the definition above.

Definition 5.8. Let M and N be T-modules. The graded stable classes of maps in $\mathrm{Ho}(\mathrm{Stmod}(T))$ is denoted by $[M,N]_*$, where $[M,N]_n := [\Sigma^n M, N] \simeq [M, \Omega^n N]$.

Remark. The definition above is proper, that is to say we have $[\Sigma^n M, N] \simeq [M,\Omega^n N]$. This is exactly what it means for the model category $\mathrm{Stmod}(T)$ to be a stable model category.

Definition 5.9. Let M and N be two T-modules. We say that M is a weak generator of $\mathrm{Stmod}(T)$ if $[M,N]_* = 0$ implies that N is weakly equivalent to $*$ for any T-module N.

Lemma 5.10. Let M be a T-module. If M is stably equivalent to some finitely generated module $\tilde{M}$, then M is automatically compact in $\mathrm{Stmod}(T)$.

Before we provide a proof for Lemma 5.10 we present some results that will be needed. We first recall some basic definitions and then give some properties for finitely generated and finitely presented modules.

Definition 5.11. Let T be a ring and M an T-module. M is said to be *finitely generated* or simply *finite* if there exists a finite subset $\{m_1, \ldots, m_n\} \subset M$ such that every element $m \in M$ is an T-linear combination of elements m_i. This is equivalent to the existence of a surjective T-module homomorphism $T^n \to M$ for some $n \in \mathbb{N}$.

Similarly, M is said to be *finitely presented* if there exists an exact sequence $T^m \to T^n \to M \to 0$ for some $n, m \in \mathbb{N}$.

Lemma 5.12. Let T be a ring and M a finitely generated projective T-module. Then M is finitely presented.

Proof. Since M is finitely generated, there exists a surjection $T^n \to M$. Since M is projective, the short exact sequence $0 \to \ker(T^n \to M) \to T^n \to M \to 0$ splits and therefore, $\ker(T^n \to M) \oplus M \simeq T^n$. Therefore $\ker(T^n \to M)$ is a direct summand of T^n and must therefore by finitely generated. This implies that our short exact sequence $0 \to \ker(T^n \to M) \to T^n \to M \to 0$ exhibits M as a finitely presented T-module. ∎

Proposition 5.13. Let T be a Frobenius ring. Then any finitely generated T-module is finitely presented.

Proof. Since T is a Frobenius ring, every projective module is an injective module. Thus, any direct sum of injective modules is a direct sum of projective modules, and thus itself a projective module and thus an injective module. This implies that T is Noetherian, therefore T^n is Noetherian, thus $\ker(T^n \to M)$ is finitely generated since it is a T-submodule of the Noetherian module T^n. Since $\ker(T^n \to M)$ is finitely generated, there exists a surjection $T^m \to \ker(T^n \to M)$ for some $m \in \mathbb{N}$ and thus $T^m \to T^n \to M \to 0$ is an exact sequence. This exact sequence exhibits M as a finitely presented T-module. ∎

Lemma 5.14. Let T be a ring. Suppose X is a finitely presented T-module. Then the canonical map $\varinjlim_i \mathrm{Hom}_T(X, M_i) \to \mathrm{Hom}_T(X, \varinjlim_i M_i)$ is bijective for any filtered colimit $\varinjlim_i M_i$.

Proof. Suppose X is finitely presented. Then there exists an exact sequence $T^m \to T^n \to X \to 0$. It is obvious that this means that X is finitely generated, since $\ker(X \to 0) = X$ and thus $\mathrm{im}(T^n \to X) = X$ is an epimorphism. We show that for a finitely presented module X, the map $\varinjlim_i \mathrm{Hom}_T(X, M_i) \to \mathrm{Hom}_T(X, \varinjlim_i M_i)$ is injective.

Let us first choose generators $x_1, \ldots, x_k \in X$ for X. Suppose that we have some map $f : N \to M_i$, for some i that is sent to the zero map $X \to M_i \to M := \varinjlim_i M_i$ by the map $\varinjlim_i \mathrm{Hom}_T(X, M_i) \to \mathrm{Hom}_T(X, \varinjlim_i M_i)$. Then for any $j \in \{1, \ldots, k\}$, there exists some $l \geq i$ such that the map $X \to M_j$ sends x_j to zero. Since the number of x_j's is finite, we can find some $l \geq i$ such that the map $X \to M_l$ is the zero map. Therefore $X \to M_i \to M_l$ is the zero map and thus, the map f is the zero map when considered as an element of $\varinjlim_i \mathrm{Hom}_T(X, M_i)$.

Let $M := \varinjlim_i M_i$ for some filtered colimit. Since X is finitely presented, we have an exact sequence $T^m \to T^n \to X \to 0$. Thus by exactness of the functor $\mathrm{Hom}_T(-, N)$ for any T-module N, we have an exact sequence $0 \to \mathrm{Hom}_T(X, M) \to \mathrm{Hom}_T(T^n, M) = M^n \to \mathrm{Hom}_T(T^m, M) = M^m$ and another exact sequence $0 \to \mathrm{Hom}_T(X, M_i) \to \mathrm{Hom}_T(T^n, M_i) = M_i^n \to \mathrm{Hom}_T(T^m, M_i) = M_i^m$. Taking the colimit of the second of these two exact sequences yields another exact sequence $0 \to \mathrm{Hom}_T(X, M_i) \to M^n \to M^m$. By padding the two sequences with an additional 0 on the left, we get two exact sequences $0 \to 0 \to \varinjlim_i \mathrm{Hom}_T(X, M_i) \to M^n \to M^m$ and $0 \to 0 \to \mathrm{Hom}_T(X, M) \to M^n \to M^m$. Using the 5-lemma for both sequences shows that $\varinjlim_i \mathrm{Hom}_T(X, M_i) \simeq \mathrm{Hom}_T(X, \varinjlim_i M_i)$. ∎

Proof. (of Lemma 5.10) We have already discussed how stable equivalences between modules turn into isomorphisms once we pass to the homotopy category. Let M and $\tilde{M}$ be stable equivalent, or equivalently weakly equivalent. That means there exists a weak equivalence $f : M \to \tilde{M}$. Therefore f is an isomorphism in the homotopy category. Therefore it suffices to show that finitely generated modules are compact in $\mathrm{Stmod}(T)$. By Lemma 5.14, finitely presented modules are compact and by Proposition 5.13 any finitely generated module over a Frobenius ring is finitely presented. Thus we are done. ∎

At this point we can refer to a number of papers that show that any additive, stable, combinatorial model category that has a compact weak generator is Quillen equivalent to the model category of modules over a certain dga. If this is of interest to the reader, we suggest consulting [6], [8], [20] or [21].

However, invoking such heavy machinery is somewhat unnecessary in this case. As Dugger & Shipley point out, the particular model categories $\mathcal{M}$ and $\mathcal{M}_\epsilon$ that we are interested in allow us to explicitly construct a Quillen equivalence between $\mathcal{M}$ and the model category of modules over a particular dga and a second Quillen equivalence between $\mathcal{M}_\epsilon$ and a model category of modules over a different particular dga.

Our main goal here is to move from working with modules over a Frobenius ring to differential graded modules. To this end, we want to construct a complete resolution of a module M, that is to say an acyclic $\mathbb{Z}$-graded chain complex of projective modules (which are automatically injective by virtue of working with Frobenius rings) such that there is an isomorphism between the module and the cycles in degree -1 of the complete resolution.

Suppose that we have a $P_\bullet M$ is such a complete resolution of M. M and ΩM can be considered as complexes concentrated in degree zero, which we do by taking M (respectively ΩM) in degree 0 and 0 everywhere else, with differentials being given by the 0 map. Since $P_\bullet M$ is a complete resolution of M, we have a natural isomorphism $Z_{-1}(P_\bullet M) \to M$, where $Z_{-1}(P_\bullet M)$ are the cycles in degree -1 of $P_\bullet M$. Therefore we have a canonical map $\pi : P_\bullet M \to M$ given by the composition of $P_0 M \to Z_{-1}(P_\bullet M)$ and $Z_{-1}(P_\bullet M) \to M$, where the first map is a projection and the second map is an isomorphism. This allows us to lift the map $P(M) \to M$ to the map $P(M) \to P_0 M$. Since we defined ΩM as the kernel of $P(M) \to M$, this lifting is actually a map of complexes $i : \Omega M \to P_\bullet M$. This lifting may not be canonical, but it is always so up to chain homotopy, which is enough in our case. Therefore, we can always go from ΩM to M via a complete resolution $P_\bullet M$ of M and this map is canonical up to chain homotopy.

We now construct one complete resolution $P_\bullet M$ of M. We define $P_n M := I(\Sigma^{-(n+1)})$ for $n < 0$ and $P_n M := P(\Omega^n M)$ for $n \geq 0$, with differentials $P_k \to P_{k-1}$ defined by the composition $P(\Omega^k M) \to \Omega^k M \to P(\Omega^{k-1} M)$ for $k - 1 > 0$, the composition $P(\Omega M) \to \Omega M \to P(M)$ for $P_1 M \to P_0 M$, the composition $P(M) \to M \to I(M)$ for $P_0 M \to P_{-1} M$ and the composition $I(\Sigma^{-(k+1)}) \to \Sigma^{-k} M \to I(\Sigma^{-k} M)$ for $P_k \to P_{k-1}$ for $k < 0$. This is summarized in the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] \arrow[d] & P(\Omega M) \arrow[r] \arrow[d] & P(M) \arrow[r] \arrow[d] & I(M) \arrow[r] \arrow[d] & I(\Sigma M) \arrow[r] \arrow[d] & \dots \\
\Omega^2 M \arrow[ur] & \Omega M \arrow[ur] & M \arrow[ur] & \Sigma M \arrow[ur] & \Sigma^2 M \arrow[ur] &
\end{tikzcd}$$

This gives us a complete resolution $P_\bullet M$.

Definition 5.15. Let Ch_T be the category of $\mathbb{Z}$-graded chain complexes of T-modules, $(X_\bullet, d_X)$, $(Y_\bullet, d_Y)$ two graded chain complexes of Ch_T and let $\mathrm{Hom}(X_\bullet, Y_\bullet)$ be the complex with $\mathrm{Hom}(X_\bullet, Y_\bullet)_n := \prod_k \mathrm{Hom}_{\mathrm{Mod}(T)}(X_k, Y_{n+k})$ the degree n T-module morphisms, without taking into consideration the differentials of the two chain complexes. Given $(f_k) \in \mathrm{Hom}(X_\bullet, Y_\bullet)_n$, we define $df \in \mathrm{Hom}(X_\bullet, Y_\bullet)_{n-1}$ as $\prod_k d_Y f_k + (-1)^{n+1} f_{k-1} d_X$.

It is clear that $\mathrm{Hom}(X_\bullet, X_\bullet)$ is a differential graded algebra over T.

Remark. For a general reference on differential graded algebras and differential graded modules over dgas, we refer to [4].

Definition 5.16. We define the endomorphism dga of M as $\mathcal{E}_M = \mathrm{Hom}(P_\bullet M, P_\bullet M)$.

There is a model category $\mathrm{Mod} - \mathcal{E}_M$ that is the category of right differential graded modules over the dga $\mathcal{E}_M$ with weak equivalences given by quasi-isomorphisms and fibrations by surjections. While we will not explicitly present this model structure, it seems worthwhile pointing out that this model structure is very comparable to the model category $\mathrm{Stmod}(T)$. For a full reference we recommend [2], wherein various model structures for differential graded modules over differential graded algebras are considered.

Lemma 5.17 ([7], Lemma 3.6)**.** Let M and N be two T-modules and $P_\bullet M$ a complete resolution of M. Then

- there are isomorphisms $H_k \mathrm{Hom}(P_\bullet M, N) \simeq [M, N]_k$ for all $k \in \mathbb{Z}$ and these isomorphisms are natural in N,
- there are isomorphisms $H_k \mathrm{Hom}(N, P_\bullet M) \simeq [N, \Omega M]_k$ for all $k \in \mathbb{Z}$ and these isomorphisms are natural in N,
- the map $\pi_* : \mathrm{Hom}(P_\bullet M, P_\bullet M) \to \mathrm{Hom}(P_\bullet M, M)$, which is induced by the map of complexes $\pi : P_\bullet M \to M$, is a quasi-isomorphism,
- the map $i_* : \mathrm{Hom}(P_\bullet M, P_\bullet M) \to \mathrm{Hom}(\Omega M, P_\bullet M)$, which is induced by the map of complexes $i : \Omega M \to P_\bullet M$, is a quasi-isomorphism.

Theorem 5.18 ([7], Theorem 3.5)**.** Let M be a compact weak generator of $\mathrm{Stmod}(T)$. Then there exists a Quillen equivalence between $\mathrm{Mod} - \mathcal{E}_M$ and $\mathrm{Stmod}(T)$, with right adjoint given by $\mathrm{Hom}(P_\bullet M, \cdot) : \mathrm{Stmod}(T) \to \mathrm{Mod} - \mathcal{E}_M$.

The Quillen equivalence in 5.18 can be explicitly split into the following two Quillen equivalences

$$\mathrm{Mod} - \mathcal{E}_M \underset{\mathrm{Hom}(P_\bullet M, -)}{\overset{- \otimes_{\mathcal{E}_M} P_\bullet M}{\rightleftarrows}} \mathrm{Ch}_T \underset{i_0}{\overset{c_0}{\rightleftarrows}} \mathrm{Stmod}(T),$$

where i_0 sends modules N to the chain complex with N concentrated in degree 0 and c_0 sends a chain complex P to $P_0/\mathrm{im}(P_1)$.

5.2.2 Existence of compact generators

The second step is showing that the two stable module categories $\mathcal{M}$ and $\mathcal{M}_\epsilon$ have compact generators. Additionally the actual model category of dg-modules over some dga is made more precise by giving the actual dga for which the Quillen equivalence holds for both $\mathcal{M}$ and $\mathcal{M}_\epsilon$.

Now we turn our attention back to $R := \mathbb{Z}/p^2$ and $R_\epsilon := ((\mathbb{Z}/p)[\epsilon])/(\epsilon^2)$. Before we work with those two rings, we need an additional definition.

Definition 5.19. Let C be an abelian category and $\mathcal{D}$ a full subcategory of C. We say that $\mathcal{D}$ is a localizing subcategory of C if there exists an exact functor $F : C \to \tilde{C}$ with a right adjoint $G : \tilde{C} \to C$ that is fully faithful and such that $\mathcal{D}$ is given by the kernel of the functor F, that is to say that objects of $\mathcal{D}$ are exactly objects X of C for which we have $F(X) = 0$.

Proposition 5.20. $\mathbb{Z}/p$ is a compact generator for $\mathrm{Stmod}(R)$ and $\mathrm{Stmod}(R_\epsilon)$.

Proof. By virtue of being finitely generated, Lemma 5.10 guarantees that $\mathbb{Z}/p$ is compact.

To show that it is a generator, we make use of a particular characterization of compact generators. For $\mathbb{Z}/p$ to be a compact generator is equivalent to every localizing subcategory that contains $\mathbb{Z}/p$ being the full category. This characterization of compact generators can be found in [20], Lemma 2.2.1.

Suppose we are working with a localizing subcategory of $\mathrm{Ho}(\mathrm{Stmod}(R))$ that contains $\mathbb{Z}/p$. Since this implies that there exists an exact functor F that exhibits the localizing subcategory as its kernel, the exact sequence $0 \to \mathbb{Z}/p \to R \to \mathbb{Z}/p \to 0$ is sent to $0 \to 0 \to F(R) \to 0 \to 0$. By exactness of the functor this new sequence is exact as well, which implies that $F(R) = 0$ and thus R is in the kernel of F and therefore in the localizing subcategory.

Therefore all free modules are contained in the localizing subcategory and so are all $\mathbb{Z}/p$-vector spaces. By Lemma 5.1 any module over R is isomorphic to a direct sum of a $\mathbb{Z}/p$-vector space and a free module. This means that all modules are contained in the localizing subcategory and therefore $\mathbb{Z}/p$ is indeed a compact generator for $\mathrm{Stmod}(R)$.

The same argument holds for R_ϵ. ■

As mentioned earlier, it is unnecessary to appeal to results from [6], [8], [20], [21], since we can directly identify the endomorphism dga associated to the generator $\mathbb{Z}/p$. This is exactly the direction Dugger & Shipley take in [7].

Proposition 5.21 ([7], Proposition 4.2)**.** The dga $\mathcal{E}_{\mathbb{Z}/p}$ in $\mathrm{Stmod}(R)$ is quasi-isomorphic to the dga A generated over $\mathbb{Z}$ by e and x in degree 1 and y in degree -1 under the relations $e^2 = 0$, $ex + xe = x^2$, $xy = yx = 1$ and differential $de = p$, $dx = 0$ and $dy = 0$, that is to say that

$$A = \mathbb{Z}\langle e, x, y\rangle/(e^2 = 0, ex + xe = x^2, xy = yx = 1, de = p, dx = 0, dy = 0),$$

with $|e| = |x| = 1$ and $|y| = -1$.

As for R, we can explicitly identify the endomorphism dga associated to the generator $\mathbb{Z}/p$ in $\mathrm{Stmod}(R_\epsilon)$.

Proposition 5.22 ([7], Proposition 4.3)**.** The dga $\mathcal{E}_{\mathbb{Z}/p}$ in $\mathrm{Stmod}(R_\epsilon)$ is quasi-isomorphic to the formal dga $A_\epsilon = \mathbb{Z}/p[x, y](xy - 1)$ with trivial differential, $|x| = 1$ and $|y| = -1$.

Proposition 5.21 follows almost directly from computations of $\mathrm{End}(\mathbb{Z}/p^2)_n$, which is defined to be $\mathrm{Hom}(\mathbb{Z}/p^2, \mathbb{Z}/p^2)_n$, where $\mathbb{Z}/p^2$ is considered as a chain complex consisting of $\mathbb{Z}/p^2$ in every dimension and the differential is given by multiplication by p. The reason that we can identify $\mathcal{E}_{\mathbb{Z}/p}$ up to quasi-isomorphism by computing $\mathrm{End}(\mathbb{Z}/p^2)_n$ is because $\mathbb{Z}/p^2$ is a complete resolution of $\mathbb{Z}/p$ and thus, by earlier remarks, we know that the dga $\mathcal{E}_{\mathbb{Z}/p}$ is quasi-isomorphic to $\mathrm{End}(\mathbb{Z}/p^2)$.

Likewise, if we consider $((\mathbb{Z}/p)[\epsilon])/(\epsilon^2)$ as a chain complex with $((\mathbb{Z}/p)[\epsilon])/(\epsilon^2)$ in every dimension and the differentials being given by multiplication by ϵ, we have a complete resolution of $\mathbb{Z}/p$ and thus a quasi-isomorphism between $\mathrm{End}(((\mathbb{Z}/p)[\epsilon])/(\epsilon^2))$ and $\mathcal{E}_{\mathbb{Z}/p}$. As was the case for $\mathbb{Z}/p^2$, computations of $\mathrm{End}(((\mathbb{Z}/p)[\epsilon])/(\epsilon^2))_n$ allow us to identify $\mathcal{E}_{\mathbb{Z}/p}$ up to quasi-isomorphism.

5.2.3 Necessary conditions for a Quillen equivalence

The third step consists of showing that there exists a necessary condition for $\mathcal{M}$ and $\mathcal{M}_\epsilon$ to be Quillen equivalent. In particular, it is shown that $\mathrm{Mod}-A$ and $\mathrm{Mod}-A_\epsilon$, which are the model categories of dg-modules over dgas that are Quillen equivalent to $\mathcal{M}$ and $\mathcal{M}_\epsilon$ respectively have to be Quillen equivalent for $\mathcal{M}$ and $\mathcal{M}_\epsilon$ to be Quillen equivalent.

Having established that a Quillen equivalence between $\mathcal{M}$ and $\mathcal{M}_\epsilon$ requires a Quillen equivalence between $\mathrm{Mod}-A$ and $\mathrm{Mod}-A_\epsilon$ reduces the problem to showing that $\mathrm{Mod}-A$ and $\mathrm{Mod}-A_\epsilon$ are not Quillen equivalent.

In order to show a lack of such a Quillen equivalence, further simplifications can be made by showing that if there were a chain of Quillen equivalences then A would have to be taken to A_ϵ in the derived equivalence of homotopy categories associated to the model categories.

Showing that there does not exist any Quillen equivalence that takes A to A_ϵ in the equivalence of homotopy categories is rather difficult without the help of additional machinery. To this end it can be shown that $\mathrm{Mod}-A$ and $\mathrm{Mod}-A_\epsilon$ are stable combinatorial model categories, which implies the existence of an associated homotopy endomorphism ring spectrum by [6]. By the necessary condition that the dga A has to be taken to A_ϵ in the derived equivalence of homotopy categories and [6], Corollary 1.4, it follows that for there to be a chain of Quillen equivalences between $\mathrm{Mod}-A$ and $\mathrm{Mod}-A_\epsilon$ there must be an isomorphism $\mathrm{hEnd}(A) \simeq \mathrm{hEnd}(A_\epsilon)$ between the two homotopy endomorphism ring spectra associated to A and A_ϵ.

Schwede & Shipley showed that two quasi-isomorphic dgas have Quillen equivalent model categories, see [19] Theorem 4.3. Using their result and Theorem 5.18 we establish a Quillen equivalence between $\mathrm{Stmod}(R)$ and $\mathrm{Mod}-A$ and another Quillen equivalence between $\mathrm{Stmod}(R_\epsilon)$ and $\mathrm{Mod}-A_\epsilon$.

Proposition 5.23. Suppose there exists a zig-zag of Quillen equivalences between $\mathrm{Stmod}(R)$ and $\mathrm{Stmod}(R_\epsilon)$. Then the derived equivalence of homotopy categories between the two homotopy categories of the two model categories maps the module $\mathbb{Z}/p \in \mathrm{Ho}(\mathcal{M})$ to an object of $\mathrm{Ho}(\mathcal{M}_\epsilon)$ that is isomorphic to the module $\mathbb{Z}/p \in \mathrm{Ho}(\mathcal{M}_\epsilon)$

Proof. We have seen before that the homotopy category of the two model categories $\mathrm{Stmod}(R)$ and $\mathrm{Stmod}(R_\epsilon)$ are isomorphic to the category of $\mathbb{Z}/p$-vector spaces.

We show that $\mathbb{Z}/p$ is mapped to an object isomorphic to $\mathbb{Z}/p$ by looking at the set of endomorphisms of $\mathbb{Z}/p$. Let $f : \mathbb{Z}/p \to \mathbb{Z}/p$ be a $\mathbb{Z}/p$-module endomorphism of $\mathbb{Z}/p$. For any $x \in \mathbb{Z}/p$ we have $f(x) = f(x \cdot 1) = xf(1)$ and thus every such endomorphism is nothing more than multiplication by $f(1)$. Therefore there are p distinct $\mathbb{Z}/p$-module endomorphisms of $\mathbb{Z}/p$.

For any free module $M = (\mathbb{Z}/p)^n$ of rank n we have a natural identification of the endomorphism algebra of M with n-by-n matrices with entries in $\mathbb{Z}/p$.

If M is an infinite dimensional vector space over $\mathbb{Z}/p$, we choose a basis. Any function from this basis to M is an endomorphism, thus the number of endomorphisms cannot be finite. Thus an infinite dimensional vector space M over $\mathbb{Z}/p$ cannot have a finite number of endomorphisms.

Therefore, the only vector space over $\mathbb{Z}/p$ with exactly p endomorphisms is a free module that is isomorphic to $\mathbb{Z}/p$. ∎

We know that the left adjoint in the Quillen equivalence $\mathrm{Mod} - \mathcal{E}_{\mathbb{Z}/p} \rightleftarrows \mathrm{Stmod}(R)$ sends $\mathcal{E}_{\mathbb{Z}/p}$ to $\mathbb{Z}/p$ and likewise for R_ϵ.

Thus, by the proposition just above, by virtue of $\mathcal{E}_{\mathbb{Z}/p}$ in $\mathrm{Stmod}(R)$ being quasi-isomorphic to A and $\mathcal{E}_{\mathbb{Z}/p}$ being quasi-isomorphic to A_ϵ in $\mathrm{Stmod}(R_\epsilon)$, we know that a potential zig-zag of Quillen equivalences between $\mathrm{Stmod}(R)$ and $\mathrm{Stmod}(R_\epsilon)$ contains a map that under the derived equivalence of homotopy categories will take A to A_ϵ, since otherwise $\mathbb{Z}/p$ would not be taken to an object isomorphic to $\mathbb{Z}/p$ in the homotopy category associated to the model categories.

5.2.4 Topological non-equivalence of dgas

As discussed in the previous step, a Quillen equivalence between $\mathrm{Mod} - A$ and $\mathrm{Mod} - A_\epsilon$ must take A to A_ϵ in the derived equivalence of homotopy categories and additionally, we would have an isomorphism of ring spectra $\mathrm{hEnd}(A) \simeq \mathrm{hEnd}(A_\epsilon)$. To finish our initial goal of showing that $\mathcal{M}$ and $\mathcal{M}_\epsilon$ are not Quillen equivalent, we only require one last proposition.

Proposition 5.24 ([7], Proposition 4.7)**.** *A and A_ϵ are not topologically equivalent.*

Since the model categories $\mathrm{Mod}-A$ and $\mathrm{Mod}-A_\epsilon$ are $\mathrm{Ch}(\mathbb{Z})$-model categories, which is to say that they are $\mathrm{Ch}(\mathbb{Z})$-enriched, tensored and cotensored, where the tensoring is given by tensor product coming from the enrichment and the cotensoring by the internal hom, and therefore additive model categories in the language of [8]. By [8], Proposition 1.5 and Proposition 1.7, there exists a weak equivalence from the homotopy endomorphism ring spectra $\mathrm{hEnd}(A)$ and $\mathrm{hEnd}(A_\epsilon)$ to their respective Eilenberg-MacLane ring spectra. Since the endomorphism dgas of A and A_ϵ are given by A and A_ϵ themselves, a Quillen equivalence of $\mathrm{Mod} - A$ and $\mathrm{Mod} - A_\epsilon$ would induce a weak equivalence between the Eilenberg-MacLane ring spectra of A and A_ϵ. This is the meaning that Dugger & Shipley give to the notion of topological equivalence of dgas. Two dgas are topologically equivalent if their corresponding Eilenberg-MacLane ring spectra are isomorphic. The question of topological equivalence of dgas is given a detailed treatment in [9]. One interesting result is that two non-quasi-isomorphic dgas can give rise to topologically equivalent dgas. On the other hand, quasi-isomorphic dgas always give rise to topologically equivalent Eilenberg-MacLane ring spectra. Thus, as alluded to earlier, Proposition 5.24 is the last piece that completes the proof that $\mathcal{M}$ and $\mathcal{M}_\epsilon$ are not Quillen equivalent.

§6 Appendix

§6.1 Category theory

Definition 6.1. A *monoidal category* is a category C along with

1. a functor $\otimes : C \times C \to C$, called the *tensor product*,
2. an object $1 \in C$, called the *unit object* or *tensor unit*,
3. a natural isomorphism $a : ((-) \otimes (-)) \otimes (-) \to (-) \otimes ((-) \otimes (-))$, with components $a_{XYZ} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, called the *associator*,
4. a natural isomorphism $\lambda : (1 \otimes (-)) \to (-)$, with components of the form $\lambda_X : 1 \otimes X \to X$, called the *left unitor*,
5. a natural isomorphism $\rho : (-) \otimes 1 \to (-)$, with components of the form $\rho_X : X \otimes 1 \to X$, called the *right unitor*,

such that the following conditions are satisfied:

1. the so-called *triangle identity* holds, that is to say that the following diagram is commutative for any choice of objects X and Y:

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr, "a_{X1Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & & X \otimes (1 \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

2. the so-called *pentagon identity* holds, that is to say that the following diagram is commutative for any choice of objects X, Y, Z, W:

$$\begin{tikzcd}
& ((X \otimes Y) \otimes Z) \otimes W \arrow[dl, "a_{XYZ} \otimes \mathrm{id}_W"'] \arrow[dr, "a_{(X \otimes Y)ZW}"] & \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[d, "a_{X(Y \otimes Z)W}"'] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[d, "a_{XY(Z \otimes W)}"] \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "\mathrm{id}_X \otimes a_{YZW}"'] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

Definition 6.2. Let C be a monoidal category, with tensor product $\otimes$, tensor unit 1, associator a, left unitor λ and right unitor ρ. A small C*-category*, or C*-enriched category*, is a small category $\mathcal{D}$ along with

1. an object $\mathcal{D}(X,Y)$ of C, called the *hom-object* of the ordered pair (X,Y) of objects of $\mathcal{D}$,
2. a morphism $\circ_{XYZ} : \mathcal{D}(Y,Z) \otimes \mathcal{D}(X,Y) \to \mathcal{D}(X,Z)$ of C, called the *composition morphism*, for any ordered triple (X,Y,Z) of objects of $\mathcal{D}$,
3. a morphism $j_X : 1 \to \mathcal{D}(X,X)$ of C, called the *identity element*, for any object X of $\mathcal{D}$,

such that for all X, Y, Z, W objects of $\mathcal{D}$, the following diagrams commute

$$\begin{tikzcd}
& (\mathcal{D}(Z,W)\otimes\mathcal{D}(Y,Z))\otimes\mathcal{D}(X,Y) \arrow[dl, "a_{\mathcal{D}(Z,W)\mathcal{D}(Y,Z)\mathcal{D}(X,Y)}"'] \arrow[dr, "\circ_{YZW}\otimes\mathrm{id}_{\mathcal{D}(X,Y)}"] & \\
\mathcal{D}(Z,W)\otimes(\mathcal{D}(Y,Z)\otimes\mathcal{D}(X,Y)) \arrow[d, "\mathrm{id}_{\mathcal{D}(Z,W)}\otimes\circ_{XYZ}"'] & & \mathcal{D}(Y,W)\otimes\mathcal{D}(X,Y) \arrow[d, "\circ_{XYZ}"] \\
\mathcal{D}(Z,W)\otimes\mathcal{D}(X,Z) \arrow[rr, "\circ_{XZW}"'] & & \mathcal{D}(X,W)
\end{tikzcd}$$

$$\begin{tikzcd}
1\otimes\mathcal{D}(X,Y) \arrow[d, "j_Y\otimes\mathrm{id}_{\mathcal{D}(X,Y)}"'] \arrow[dr, "\lambda"] & & \mathcal{D}(X,Y)\otimes 1 \arrow[d, "\mathrm{id}_{\mathcal{D}(X,Y)}\otimes j_X"] \arrow[dl, "\rho"'] \\
\mathcal{D}(Y,Y)\otimes\mathcal{D}(X,Y) \arrow[r, "\circ_{XYY}"'] & \mathcal{D}(X,Y) & \mathcal{D}(X,Y)\otimes\mathcal{D}(X,X) \arrow[l, "\circ_{XXY}"]
\end{tikzcd}$$

where the first diagram exhibits the associativity of composition in $\mathcal{D}$ and the second diagram exhibits the unitality of the composition in $\mathcal{D}$.

Definition 6.3. The *category of abelian groups* Ab is the category whose objects are abelian groups and morphisms are group homomorphisms of abelian categories. The category of abelian groups is a prime example of a monoidal category. In fact, Ab is a symmetric monoidal category, where symmetric signifies that for any two abelian groups G and H there is a natural isomorphism between $G \otimes H$ and $H \otimes G$.

Definition 6.4. Let C and $\mathcal{D}$ be two finitely complete categories. A functor $F : C \to \mathcal{D}$ is *left exact* if it preserves finite limits.

Dually, a functor $F : C \to \mathcal{D}$ between two finitely cocomplete categories is called *right exact* if it preserves finite colimits.

Definition 6.5. A *pre-abelian category* is a finitely bicomplete Ab-enriched category, that is to say an Ab-enriched category that is finitely complete and finitely cocomplete or equivalently has all finite limits and finite colimits.

Definition 6.6. An *abelian category* is a pre-abelian category such that the following equivalent conditions are satisfied:

1. for any morphism f, the canonical morphism $\mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f))$ is an isomorphism,

2. every monomorphism can be exhibited as the kernel of a morphism and every epimorphism can be exhibited as the cokernel of a morphism.

Remark. One of the main advantages of working with abelian categories is that any morphism $f : X \to Y$ factors through its own image, that is to say we can factor f as $X \to \mathrm{im}(f) \to Y$, such that this factorization consists of an epimorphism followed by a monomorphism and this factorization is unique up to a unique isomorphism.

Proposition 6.7. Let C be an abelian category and $0 \to X \to Y \to Z \to 0$ an exact sequence in C. A functor $F : C \to \mathcal{D}$, with $\mathcal{D}$ an abelian category, is

- left exact if and only if it preserves direct sums and kernels,

- right exact if and only if it preserves direct sums and cokernels.

Moreover,

- if F is left exact, then
$$0 \to F(X) \to F(Y) \to F(Z)$$
is an exact sequence in $\mathcal{D}$,
- if R is right exact, then
$$F(X) \to F(Y) \to F(Z) \to 0$$
is an exact sequence in $\mathcal{D}$,
- if F is exact, then
$$0 \to F(X) \to F(Y) \to F(Z) \to 0$$
is an exact sequence in $\mathcal{D}$ and F preserves chain homology.

Definition 6.8. A category C is said to be *locally presentable* if

1. C is a locally small category,
2. C is cocomplete,
3. there exists a regular cardinal α and a proper set S consisting of objects of C, which are all α-small and generate C under α-filtered colimits,
4. all objects of C are small.

Definition 6.9. An *additive category* is an Ab-enriched category that admits finite coproducts.

Definition 6.10. A strict 2-category C consists of

- a collection of objects,
- for any pair of objects $X, Y \in C$, a category $\hom_C(X, Y)$,whose objects are called 1-morphisms from X to Y and whose morphisms are given by 2-morphisms between 1-morphisms, that is for $f, g : X \to Y$ two 1-morphisms, a 2-morphism is a map $f \Rightarrow g$,
- for any triple of objects $X, Y, Z \in C$, a composition functor
$$\circ : \hom_C(Y, Z) \times \hom_C(X, Y) \to \hom_C(X, Z),$$
- for every object $X \in C$, an identity 1-morphism $\mathrm{id}_X \in \mathrm{Hom}_C(X, X)$,

such that the following conditions are satisfied:

1. for any object $X \in C$, the identity 1-morphism id_X is a unit for right and left composition, that is to say that the following two functors are equal to the identity
$$\begin{aligned} \hom_C(X, Y) \to \hom_C(X, Y) &\quad f \mapsto f \circ \mathrm{id}_X, \\ \hom_C(Y, X) \to \hom_C(Y, X) &\quad g \mapsto \mathrm{id}_X \circ g, \end{aligned}$$

2. the composition law of C is associative, that is to say that for $X, Y, Z, W \in C$, the following diagram commutes

$$\begin{tikzcd}
\hom_C(Y,Z) \times \hom_C(X,Y) \times \hom_C(W,X) \arrow[r, "\mathrm{id}\times\circ"] \arrow[d, "\circ\times\mathrm{id}"'] & \hom_C(Y,Z) \times \hom_C(W,Y) \arrow[d, "\circ"] \\
\hom_C(X,Z) \times \hom_C(W,X) \arrow[r, "\circ"'] & \hom_C(W,Z)
\end{tikzcd}$$

The composition of two 2-morphisms $\alpha : f \Rightarrow g$, $\beta : g \Rightarrow h$, where $f, g, h : X \to Y$ for some $X, Y \in C$ is the composition $\beta \circ \alpha : f \Rightarrow h$ is called the *vertical composition of* 2*-morphisms* and the composition of two 2-morphisms $\alpha : f \Rightarrow g$, $\beta : h \Rightarrow k$ where $f, g : X \to Y$ and $h, k : Y \to Z$, via the composition functor $\circ$, is called the *horizontal composition of* 2*-morphisms.*

Definition 6.11. We define the strict 2-category of small categories Cat as consisting of

- small categories as objects,
- functors between small categories as 1-morphisms,
- natural transformations between functors as 2-morphisms.

The vertical composition of 2-morphisms is given by the obvious composition of the components of natural transformations and the horizontal composition is given by the Godement product, that is given two natural transformations $\alpha : F_1 \to G_1 : C_1 \to C_2$ and $\beta : F_2 \to G_2 : C_2 \to C_3$, where F_1, G_1 are functors from the category C_1 to the category C_2 and F_2, G_2 are functors from the category C_2 to the category C_3, their Godement product $\beta \circ \alpha : F_2 \circ F_1 \to G_2 \circ G_1 : C_1 \to C_3$ is again a natural transformation and can be fully defined by its component maps. Let $X \in C_1$, then we define the component map of the natural transformation $(\beta \circ \alpha)_X := \beta_{G_1(X)} \circ F_2(\alpha_X)$ or equivalently $(\beta \circ \alpha)_X := G_2(\alpha_X) \circ \beta_{F_1(X)}$.

Definition 6.12. Suppose C is a category that has pullbacks. We say that $\mathcal{D}$ is a *category internal to* C if it consists of

- an object of objects $\mathcal{D}_0 \in C$,
- an object of morphisms $\mathcal{D}_1 \in C$,
- a source and a target morphism $s, t : \mathcal{D}_1 \to \mathcal{D}_0$,
- a morphism that assigns the identity to each object $e : \mathcal{D}_0 \to \mathcal{D}_1$,
- a morphism that allows for composition $c : \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \to \mathcal{D}_1$,

such that we have the following commutative diagrams

$$\begin{tikzcd}
\mathcal{D}_0 \arrow[r, "e"] \arrow[dr, "\mathrm{id}_{\mathcal{D}_0}"'] & \mathcal{D}_1 \arrow[d, "s"] \\
& \mathcal{D}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}_0 \arrow[r, "e"] \arrow[dr, "\mathrm{id}_{\mathcal{D}_0}"'] & \mathcal{D}_1 \arrow[d, "t"] \\
& \mathcal{D}_0
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \arrow[r, "c"] \arrow[d, "p_1"'] & \mathcal{D}_1 \arrow[d, "s"] \\
\mathcal{D}_1 \arrow[r, "s"'] & \mathcal{D}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \arrow[r, "c"] \arrow[d, "p_2"'] & \mathcal{D}_1 \arrow[d, "t"] \\
\mathcal{D}_1 \arrow[r, "t"'] & \mathcal{D}_0
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \arrow[r, "(c,\mathrm{id}_{\mathcal{D}_1})\circ((p_1,p_2),p_3)"] \arrow[d, "(\mathrm{id}_{\mathcal{D}_1},c)\circ(p_1,(p_2,p_3))"'] & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \arrow[d, "c"] \\
\mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \arrow[r, "c"'] & \mathcal{D}_1
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D}_0 \times_{\mathcal{D}_0} \mathcal{D}_1 \arrow[r, "e\times_{\mathcal{D}_0}\mathrm{id}_{\mathcal{D}_1}"] \arrow[dr, "p_2"'] & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \arrow[d, "c"] & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_0 \arrow[l, "\mathrm{id}_{\mathcal{D}_1}\times_{\mathcal{D}_0} e"'] \arrow[dl, "p_1"] \\
& \mathcal{D}_1 &
\end{tikzcd}$$

where $\mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1$ is the pullback of s and t and $\mathcal{D}_0 \times_{\mathcal{D}_0} \mathcal{D}_1$ is the pullback of $\mathrm{id}_{\mathcal{D}_0}$ and t and $\mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_0$ is the pullback of s and $\mathrm{id}_{\mathcal{D}_0}$.

Definition 6.13. A *double category* $\mathcal{D}$ is an internal category in the 2-category of small categories Cat.

Definition 6.14. Let $\mathcal{C}$ be a locally small category. The *Yoneda embedding* is the functor $h : \mathcal{C} \to \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ defined by $h(Z) := \mathrm{Hom}_{\mathcal{C}}(-, Z)$ for any object Z of $\mathcal{C}$ and by mapping any morphism $f : X \to Y$ of $\mathcal{C}$ to the morphism $h(f) := \mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{\mathcal{C}}(-, Y)$, which is defined by composition with f, that is to say the natural transformation $\mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathrm{Hom}_{\mathcal{C}}(-, Y)$ is defined component wise by $h(f)_Z = \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ by $h(f)_Z(g) = f \circ g$ for any $g : Z \to X$, for any object Z of $\mathcal{C}$.

Lemma 6.15 (Yoneda Lemma)**.** Let $\mathcal{C}$ be a locally small category. For any functor $F \in \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, there exists a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})}(h(Y), F) \simeq F(Y),$$

where h is the Yoneda embedding and Y any object of $\mathcal{C}$.

Remark. The Yoneda lemma is one of the most fundamental results in category theory. As such, proofs can be found in various references for category theory, such as [5] or [14].

§6.2 Simplicial sets

We will provide basic definitions and basic results concerning simplicial sets, since simplicial sets come up occasionally in this thesis. When it comes to references for simplicial sets, we highly recommend [15] and [5].

Definition 6.16. The *simplex category* Δ is the category whose

- objects are linearly ordered sets of the form $[n] := \{0, 1, \dots, n\}$ for $n \geq 0$,
- morphisms are given by nondecreasing functions.

Definition 6.17. Let $\mathcal{C}$ be a category. A *simplicial object of* $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. Dually, a *cosimplicial object of* $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$.

Equivalently, we can define a simplicial object of a category $\mathcal{C}$ as a $\mathcal{C}$-valued presheaf over the simplex category Δ.

Definition 6.18. A *simplicial set* is a simplicial object of Set, that is to say a presheaf over the simplex category Δ.

A simplicial set $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$ maps $[n]$ to a set $X([n])$, for each $n \geq 0$. We call elements of $X([n])$ the n-simplices of X.

Definition 6.19. The *category of simplicial sets* SSet is defined as the functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$.

Definition 6.20. Let $n \geq 0$. The *standard n-simplex* is the simplicial set given by

$$([m] \in \Delta) \mapsto \mathrm{Hom}_\Delta([m], [n])$$

Remark. Suppose $S : \Delta^{\mathrm{op}} \to \mathrm{Set}$ is a simplicial set. We will indicate that S is a simplicial set by denoting it by $S_\bullet$ and define $S_n := S([n])$, that is S_n is the set of n-simplices of $S_\bullet$.

Proposition 6.21. The standard n-simplex Δ^n is a simplicial set.

Proof. Let C be a category. Recall that $\mathrm{Hom}_C(-, X) : C \to \mathrm{Set}$ is a contravariant functor for any object $X \in C$. Therefore Δ^n is a presheaf over Δ and thus a simplicial set. ∎

Proposition 6.22. For every simplicial set $X_\bullet$, there is a bijection

$$\mathrm{Hom}_{\mathrm{SSet}}(\Delta^n, X_\bullet) \simeq X_n.$$

Proof. Follows directly from the Yoneda Lemma 6.15. ∎

Remark. Proposition 6.22 allows us to identify n-simplices of any simplicial set $X_\bullet$ with maps of simplicial sets $\sigma : \Delta^n \to X_\bullet$.

In what follows, we introduce the face and degeneracy maps for simplicial sets. First, we introduce two special types of maps in the simplex category. For any integer $n \geq 1$ and $0 \leq i \leq n$, we define $\varphi_i^n : [n-1] \to [n]$ by

$$\varphi_i^n(x) = \begin{cases} x & \text{if } x < i, \\ x+1 & \text{if } x \geq i. \end{cases}$$

and for any integer $n \geq 0$ and $0 \leq i \leq n$, we define $\psi_i^n : [n+1] \to [n]$ by

$$\psi_i^n(x) = \begin{cases} x & \text{if } x \leq i, \\ x-1 & \text{if } x \geq i+1. \end{cases}$$

Since these maps are morphisms in the simplex category and we defined the n-simplex Δ^n as the functor $([m] \mapsto \mathrm{Hom}_\Delta([m], [n]))$, these maps induce maps $\partial_i^n : \Delta^{n-1} \to \Delta^n$ and $\omega_i^n : \Delta^{n+1} \to \Delta^n$.

Let $X_\bullet$ be a simplicial set. As we remarked above, n-simplices of a simplicial set can be identified with maps of simplicial sets $\Delta^n \to X_\bullet$. Thus, we have $X_n \simeq \mathrm{Hom}_{\mathrm{SSet}}(\Delta^n, X_\bullet)$ and the maps ∂_i^n and ω_i^n induce maps $\mathrm{Hom}_{\mathrm{SSet}}(\Delta^{n+1}, X_\bullet) \to \mathrm{Hom}_{\mathrm{SSet}}(\Delta^n, X_\bullet)$ and $\mathrm{Hom}_{\mathrm{SSet}}(\Delta^{n-1}, X_\bullet) \to \mathrm{Hom}_{\mathrm{SSet}}(\Delta^n, X_\bullet)$.

Definition 6.23. The *face maps* and *degeneracy maps*, with the former denoted by $s_n^i : X_n \to X_{n+1}$ for $n \geq 0$ and $0 \leq i \leq n$ and the latter denoted by $d_n^i : X_n \to X_{n-1}$ for $n \geq 1$ and $0 \leq i \leq n$, are the maps induced by the maps ω and ∂ as described above.

Definition 6.24. Let $n \geq 0$. The *boundary of* Δ^n is defined as the simplicial set $\partial\Delta^n : \Delta^{\mathrm{op}} \to \mathrm{Set}$ by the formula

$$\partial\Delta^n([m]) = \{f \in \mathrm{Hom}_\Delta([m], [n]) : f \text{ is not surjective}\}.$$

Definition 6.25. The simplicial set $\Lambda_i^n : \Delta^{op} \to \mathrm{Set}$ given by the formula

$$(\Lambda_i^n)([m]) = \{f \in \mathrm{Hom}_\Delta([m], [n]) : [n] \nsubseteq f([m]) \cup \{i\}\},$$

is called the *i-th horn in* Δ^n.

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