Test functions, mollifiers and convolution

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The material presented in this document was motivated by the desire to have a rigurous proof for a result that is a very useful tool and makes its appearance in mathematical literature frequently. One such example is the notion of weak derivatives. In particular the result that will be developed throughout this document is used to show that weak derivatives are unique, should they exist. The result that this document sets out to prove can be stated as follows.

Theorem 1. *Given an open set* $\Omega \subset \mathbb{R}^n$ *and a function* $f \in L^1_{\text{loc}}(\Omega)$ *we have that if* $\int_\Omega f\varphi dx = 0$ *for all* $\varphi \in C_0^\infty(\Omega)$*, then* $f = 0$ *almost everywhere in* Ω*.*

Proving this theorem rigurously requires quite a bit of work. This document attempts to introduce and develop everything needed for a complete proof. We first start out by introducing a relatively simple function and highlighting some of its basic properties.

Lemma 2.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

is of $C^\infty(\mathbb{R})$ *regularity.*

Proof. **Step 1.** We first show that for $x > 0$ the k-th derivative of f is of the following form.

$$f^{(k)}(x) = P(\tfrac{1}{x})f(x),$$

where $P(\frac{1}{x})$ is a polynomial in $\frac{1}{x}$.

For $k = 0$ and $k = 1$ this holds trivially, seeing as $f^{(0)}(x) = f(x)$ and $f^{(1)}(x) = \frac{1}{x^2}e^{-\frac{1}{x}} = \frac{1}{x^2}f(x)$. By hypothesis of induction, we suppose that for some $k = 1, 2, ...$ f does indeed take the above form. Computing the $k+1$-th derivative of f yields

$$\begin{aligned} f^{k+1}(x) &= -\tfrac{1}{x^2}P'(\tfrac{1}{x})f(x) + P(\tfrac{1}{x})f'(x) \\ &= \tfrac{1}{x^2}\big(P(\tfrac{1}{x}) - P'(\tfrac{1}{x})\big)f(x) \\ &= Q(\tfrac{1}{x})f(x). \end{aligned}$$

Since $Q(\frac{1}{x}) = \frac{1}{x^2}\big(P(\frac{1}{x}) - P'(\frac{1}{x})\big)$ is a polynomial in $\frac{1}{x}$ we move on to the next step.

Step 2. We now show that for any polynomial in $\frac{1}{x}$, i.e. $P(\frac{1}{x})$, we have

$$\lim_{x\to 0^+} P(\tfrac{1}{x})f(x) = 0.$$

Via series definition of the exponential, we can establish the following upper bound for f.

$$0 \leq f(x) \leq x^n n!,$$

for any $n = 0, 1, 2, ...$

Given any polynomial in $\frac{1}{x}$ $P(\frac{1}{x})$, any n bigger than the degree of P can be used to show that $\lim_{x\to 0^+} P(\frac{1}{x})f(x) = 0$.

Step 3. As a final step we show that $f^{(k)}(0) = 0$ for all $k = 0, 1, 2, ...$ This is done via induction. For $k = 0$ by definition this holds as $f^{(0)}(0) = f(0) = 0$. Suppose now that $f^{(k)}(0) = 0$ for some $k \geq 0$. By the definition of the derivative, we know that $f^{(k+1)}(0) = \lim_{t\to 0} \frac{f^{(k)}(t) - f^{(k)}(0)}{t} = \lim_{t\to 0} \frac{f^{(k)}(t)}{t}$. In the case of $t < 0$ we trivially have the desired result of $\lim_{t\to 0^-} \frac{f^{(k)}(t)}{t} = 0$. For $t > 0$ it suffices to remark that $\lim_{t\to 0^+} \frac{f^{(k)}(t)}{t} = \lim_{t\to 0^+} P(\frac{1}{t})\frac{1}{t}f(t)$. By virtue of $P(\frac{1}{t})\frac{1}{t}$ being a polynomial in $\frac{1}{t}$, the result established in Step 2 allows us to directly conclude that $\lim_{t\to 0^+} \frac{f^{(k)}(t)}{t} = 0$, which concludes the proof.

□

Lemma 3. *There exists a function $\phi : \mathbb{R}^n \to \mathbb{R}$, for which the following holds true.*

i) $\phi(x) \geq 0$ *for all* $x \in \mathbb{R}^n$,

ii) $\phi \in C^\infty(\mathbb{R}^n)$,

iii) $\text{supp } \phi = \overline{B_1(0)}$,

iv) $\int\limits_{\mathbb{R}^n} \phi \; dx = 1$.

Proof. We define $\phi(x)$ by making use of the function f introduced in Lemma 2.

$$\phi(x) := c_n f(1 - \|x\|^2),$$

where c_n is a positive constant dependent only on n, meaning the dimension of $\mathbb{R}^n$. $\|x\|^2$ is the standard euclidian norm in $\mathbb{R}^n$.

To show the first property it suffices to remark that f as defined in Lemma 2 is a positive function over $\mathbb{R}$ and since c_n is a positive constant we conclude that $\phi \geq 0$.

Proving C^∞ regularity requires a bit more work. ϕ is a compound function of the polynomial $1 - \|x\|^2$ and the function f, for which we established C^∞ regularity in Lemma 2. It suffices to use standard rules of differentiation, i.e. applying both the chain- and product-rule in multiple variables, to show the continuity of each partial derivative of ϕ.

The third property is trivial to prove. We know that $\forall x \notin \overline{B_1(0)}$ $1 - \|x\|^2 < 0$ and therefore $f(x) = 0$, which implies $x \notin \text{supp } \phi$.

Seeing as we have not made use of the constant c_n in any way, we can set define c_n such that the last property holds. ∎

Definition 4. A mollifier is a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$, i.e. $\varphi \in C^\infty(\mathbb{R}^n)$, if the following conditions hold.

i) φ is of compact support,

ii) $\int\limits_{\mathbb{R}^n} \varphi(x) dx = 1$,

iii) $\lim\limits_{\varepsilon \to 0} \varphi_\varepsilon(x) = \lim\limits_{\varepsilon \to 0} \varepsilon^{-n} \varphi(\frac{x}{\varepsilon}) = \delta(x)$,

where $\delta(x)$ is the Dirac delta function and the limit taken in the schwartz space $S(\mathbb{R}^n)$.

Remark 1. i) The Schwartz space $S(\mathbb{R}^n)$ contains the space of all test functions $C_c^\infty(\mathbb{R}^n)$,

ii) $C_c^\infty(\mathbb{R}^n)$ is also referred to as the space of bump functions.

Definition 5. A mollifier φ is a

i) positive mollifier, if $\varphi \geq 0$ in $\mathbb{R}^n$,

ii) symmetric mollifier, if $\varphi(x) = \psi(|x|)$, with $\psi \in C^\infty(\mathbb{R}_+)$.

Lemma 6. *The function ϕ, as defined in Lemma 3, is a symmetric and positive mollifier. Additionally for any $\varepsilon > 0$*

$$\int\limits_{\mathbb{R}^n} \phi_\varepsilon(x) dx = 1.$$

Proof. ϕ being a mollifier follows directly from the definition of mollifiers and the properties established for the function ϕ in Lemma 3. To show that $\int_{\mathbb{R}^n} \phi_\varepsilon(x)dx = 1$ we use a change of variable $y = \frac{x}{\varepsilon}$ in the equality $\int_{\mathbb{R}^n} \phi(x)dx = 1$, which follows from Lemma 3. □

Remark 2. The function ϕ as defined in Lemma 3 is considered the standard mollifier.

We now introduce the notion of convolution. We denote the convolution of two functions f and φ by $f * \varphi$ and define it as follows.

$$(f * \varphi)(x) := \int_{\mathbb{R}^n} f(x-y)\varphi(y)dy = \int_{\mathbb{R}^n} f(y)\varphi(x-y)dy.$$

Lemma 7. *If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\psi \in C_c(\mathbb{R}^n)$, then $f * \psi$ is continuous.*

Proof. We prove this claim by showing that for any convergent sequence $\{x_m\} \subset \mathbb{R}^n$, $x_m \to x \in \mathbb{R}^n$ the following holds.

$$\lim_{m\to\infty} (f * \psi)(x_m) = (f * \psi)(x).$$

Before we attempt to prove this, please note that $f \in L^1_{\text{loc}}$ allows f to be redefined arbitrarily on any set of measure 0 without changing $f * \psi$. Therefore we may assume that f is defined everywhere.

Since x_m converges to x, there exists $R > 0$ such that $\{x_m\} \subset B_R(x)$. Let $r > 0$ such that $\text{supp}\psi \subset B_r(0)$. By continuity of ψ, we have for all y

$$f(y)\psi(x_m - y) \to f(y)\psi(x-y).$$

On top of that for any y such that $\|y-x\| > R+r$ we have $\|y-x_m\| \geq \|y-x\| - \|x-x_m\| > r+R-R = r$, which means that $y \notin \text{supp}\psi$ and therefore $\psi(x_m - y) = 0$. This allows us to establish the following bound

$$|f(y)\psi(x_m-y)| \leq \sup(|\psi|)|f(y)|\mathcal{X}_{\overline{B_{r+R}(x)}}(y)$$

By the dominated convergence theorem we can integrate, since the function on the righthand side is integrable.

$$\begin{aligned}\lim_{m\to\infty} (f * \psi)(x_m) &= \lim_{m\to\infty} \int_{\mathbb{R}^n} f(y)\psi(x_m - y)dy \\ &= \int_{\mathbb{R}^n} \lim_{m\to\infty} f(y)\psi(x_m-y)dy \\ &= \int_{\mathbb{R}^n} f(y)\psi(x-y)dy \\ &= (f * \psi)(x).\end{aligned}$$

□

Lemma 8. *Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a function $\psi \in C_c^\infty(\mathbb{R}^n)$, the following holds*

$$\frac{\partial(f * \psi)}{\partial x_i}(x) = (f * \frac{\partial \psi}{\partial x_i})(x),$$

for any $i = 1, 2, \ldots$.

Proof. For some i we compute the following

$$\begin{aligned}\frac{\partial(f*\psi)}{\partial x_i}(x) &= \lim_{t\to 0}\tfrac{1}{t}\big((f*\psi)(x+t\cdot e_i)-(f*\psi)(x)\big)\\ &= \lim_{t\to 0}\tfrac{1}{t}\Big(\int\limits_{\mathbb{R}^n} f(y)\psi(x+t\cdot e_i)dy - \int\limits_{\mathbb{R}^n} f(y)\psi(x-y)dy\Big)\\ &= \lim_{t\to 0}\int\limits_{\mathbb{R}^n} f(y)\big(\tfrac{\psi(x+t\cdot e_i)-\psi(x-y)}{t}\big)dy.\end{aligned}$$

The mean value theorem gives us $s(t)$ such that $|s(t)|\leq|t|$ and $\frac{\partial\psi}{\partial x_i}(x-y+s(t)\cdot e_i)=\frac{\psi(x+t\cdot e_i)-\psi(x-y)}{t}$. Therefore we have

$$\begin{aligned}\tfrac{\partial(f*\psi)}{\partial x_i}(x) &= \lim_{t\to 0}\int\limits_{\mathbb{R}^n} f(y)\big(\tfrac{\psi(x+t\cdot e_i)-\psi(x-y)}{t}\big)dy\\ &= \lim_{t\to 0}\int\limits_{\mathbb{R}^n} f(y)\tfrac{\partial\psi}{\partial x_i}(x-y+s(t)\cdot e_i)dy\\ &= \lim_{t\to 0}\big(f*\tfrac{\partial\psi}{\partial x_i}\big)\big(x+s(t)\cdot e_i\big)\\ &= (f*\tfrac{\partial\psi}{\partial x_i})(x),\end{aligned}$$

with the last equality being a consequence of Lemma 7. □

Theorem 9. *If $f\in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\psi\in C_c^\infty(\mathbb{R}^n)$, then $(f*\psi)\in C^\infty(\mathbb{R}^n)$.*

Proof. Using Lemma 8 we compute $\frac{\partial(f*\psi)}{\partial x_i}(x)=(f*\frac{\partial\psi}{\partial x_i})(x)$. Since $\frac{\partial\psi}{\partial x_i}$ is $C_c(\mathbb{R}^n)$ we can apply Lemma 7 and therefore we have that $\frac{\partial(f*\psi)}{\partial x_i}$ is continuous.

Higher-order derivatives are continuous as well, which follows from repeating the above procedure. □

Lemma 10. *For any function $f\in L^1_{\text{loc}}(\mathbb{R}^n)$, we have*

$$\text{supp}(f*\phi_\varepsilon)\subset\text{supp}f+\overline{B_\epsilon(0)}==\{x+y\mid x\in\text{supp}(f),\ y\in\overline{B_\varepsilon(0)}\},$$

where ϕ is the standard mollifier.

Proof.

$$f_\varepsilon(x)=\int\limits_{\mathbb{R}^n} f(x-y)\phi_\varepsilon(y)dy=\int\limits_{B_\varepsilon(0)} f(x-y)\phi_\varepsilon(y)dy.$$

Suppose that for some x, we have $f_\varepsilon(x)=0$. Then $f(x-y)$ cannot be identically 0 for all $y\in B_\varepsilon(0)$. Therefore there exists at least one such y, for which $x-y\in\text{supp}(f)$ and thus $x=(x-y)+y\in\text{supp}(f)+B_\varepsilon(0)$. □

Theorem 11. *For any function $f\in C(\mathbb{R}^n)$*

$$f_\varepsilon=f*\phi_\varepsilon\to f,$$

converges uniformly on all compact subsets of $\mathbb{R}^n$, where ϕ_ϵ is the standard mollifier.

Proof. Let $K \subset\subset \mathbb{R}^n$, i.e. K is a compact subset of $\mathbb{R}^n$. By compactness, there exists some $r > 0$ such that $K \subset \overline{B_r(0)}$. Additionally since f is continuous, f is also absolutely continuous on the compact set $\overline{B_{r+1}(0)}$. Therefore for any $\alpha > 0$, there exists δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \alpha$.

For any $x \in K \subset \overline{B_r(0)}$ and $\varepsilon \in (0,1) \cap (0,\delta)$ we find that the following holds.

$$\begin{aligned}
|f_\varepsilon(x) - f(x)| &= \Big| \int_{\mathbb{R}^n} f(x-y)\phi_\varepsilon(y)dy - f(x) \Big| \\
&= \Big| \int_{\mathbb{R}^n} f(x-y)\phi_\varepsilon(y)dy - \int_{\mathbb{R}^n} f(x)\phi_\varepsilon(y)dy \Big| \\
&\le \int_{\mathbb{R}^n} |f(x-y) - f(x)|\phi_\varepsilon(y)dy \\
&= \int_{\overline{B_\varepsilon(0)}} |f(x-y) - f(x)|\phi_\varepsilon(y)dy \\
&< \int_{\overline{B_\varepsilon(0)}} \alpha\phi_\varepsilon(y)dy = \alpha,
\end{aligned}$$

where the last line follows from $x \in \overline{B_r(0)} \subset \overline{B_{r+1}(0)}$ and $x - y \in \overline{B_r(0)} + \overline{B_\varepsilon(0)} \subset \overline{B_{r+1}(0)}$. Since this inequality holds for any $x \in K$ we have that $f_\varepsilon \to f$ uniformly on K. And since K is any compact subset of $\mathbb{R}^n$ this completes the proof. ☐

Theorem 12. *For any* $1 \le p < \infty$, $C_c^\infty(\mathbb{R}^n)$ *is dense in* $L^p(\mathbb{R}^n)$.

Proof. As a starting point in this proof we assume that the fact that $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ is known. For further details and a detailed proof of this fact please consult [1], specifically theorem 3.14.

Let $f \in L^1(\mathbb{R}^n)$. By density of $C_c(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ we know that there exists a function $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_{L^p(\mathbb{R}^n)} \le \frac{\delta}{2}$ for any $\delta > 0$.

Now we take the convolution g_ε, which converges uniformly to g as $\varepsilon \to 0$ thanks to Theorem 11. Since g is of compact support, there exists $r > 0$ such that $\operatorname{supp}(g) \subset B_r(0)$ and by Lemma 10 we have $\operatorname{supp}(g_\varepsilon) \subset \operatorname{supp}(g) + \overline{B_\varepsilon(0)} \subset B_r(0) + \overline{B_\varepsilon(0)} = B_{r+\varepsilon}(0)$.

Simple computation gives us the following upper bound

$$\|g_\varepsilon - g\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |g_\varepsilon(x) - g(x)|^p dx = \int_{B_{r+\varepsilon}(0)} |g_\varepsilon(x) - g(x)|^p dx \le |B_{r+\varepsilon}(0)| \sup_{x \in B_{r+\varepsilon}(0)} |g_\varepsilon(x) - g(x)|^p.$$

Since $g_\varepsilon \to g$ uniformly as $\varepsilon \to 0$ we also know that $\sup|g_\varepsilon - g|^p \to 0$ as $\varepsilon \to 0$. Therefore there exists $\alpha > 0$ such that for all $\varepsilon < \alpha$ we have $\|g_\varepsilon - g\|^p_{L^p(\mathbb{R}^n)} < \frac{\delta}{2}$ and this yields

$$\|f - g_\varepsilon\|_{L^p(\mathbb{R}^n)} \le \|f - g\|_{L^p(\mathbb{R}^n)} + \|g - g_\varepsilon\|_{L^p(\mathbb{R}^n)} < \delta.$$

g_ε is of C^∞ regularity by Theorem 9 and of compact support since $\operatorname{supp}(g_\varepsilon) \subset \operatorname{supp}(g) + \overline{B_\varepsilon(x)}$. Therefore $g_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and thus $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, which completes the proof. ☐

Theorem 13. *For any function* $f \in L^p(\mathbb{R}^n)$ *with* $1 \le p < \infty$ *we have* $\|f * \phi_\varepsilon - f\|_{L^p(\mathbb{R}^n)} \to 0$ *as* $\varepsilon \to 0$ *where* ϕ_ε *is the standar mollifier.*

Proof. We have

$$|(f * \phi_\varepsilon)(x) - f(x)| \leq \int_{\mathbb{R}^n} |f(x-y) - f(x)|\phi_\varepsilon(y)dy.$$

Using Jensen's inequality for the convex function $t \mapsto t^p$ yields

$$\begin{aligned}|(f * \phi_\varepsilon)(x) - f(x)|^p &\leq \Big(\int_{\mathbb{R}^n} |f(x-y) - f(x)|\phi_\varepsilon(y)dy\Big)^p \\ &\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)|^p \phi_\varepsilon(y)dy.\end{aligned}$$

Integrating over $\mathbb{R}^n$ in x and applying Fubini's theorem shows the following.

$$\begin{aligned}\|f * \phi_\varepsilon - f\|_p^p &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p \phi_\varepsilon(y)dydx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y) - f(x)|^p dx\phi_\varepsilon(y)dy \\ &= \int_{\mathbb{R}^n} \|f^y - f\|_p^p \phi_\varepsilon(y)dy,\end{aligned}$$

where $f^y(x) = f(x-y)$. We now show that the function $g(y) = \|f^y - f\|_p^p$ is continuous and bounded and $g(0) = 0$. $g(0) = 0$ is trivial and g being bounded follows directly from

$$g(y) = \|f^y - f\|_p^p \leq (\|f^y\|_p + \|f\|_p)^p = (2\|f\|_p)^p.$$

To show that g is continuous we proceed as follows: As previously established, we can find a continuous function h with compact support such that $\|f - h\|_{L^p(\mathbb{R}^n)} < \frac{\delta}{3}$. Since h is continuous and of compact support we can find $r > 0$ such that $\text{supp}(h) \subset B_r(0)$ and h is uniformly continuous on $B_r(0)$. Thus there exists $\alpha \in (0, r)$ such that for $|s-t| < \alpha$ we have $|h(s) - h(t)| < |B_{2r}(0)|^{-\frac{1}{p}} \frac{\delta}{3}$. We define $h^s(x) = h(x-s)$ and $h^t(x) = h(x-t)$ and compute for any s and t such that $|s-t| < \alpha$

$$\|h^s - h^t\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |h(x-s) - h(x-t)|^p dx < \frac{1}{|B_{2r}(0)|}\delta^p |B_{r+\alpha}(s)| < \Big(\frac{\delta}{3}\Big)^p.$$

For all s, t such that $|s-t| < \alpha$,

$$\begin{aligned}\|f^s - f^t\|_{L^p(\mathbb{R}^n)} &\leq \|f^s - h^s\|_{L^p(\mathbb{R}^n)} + \|h^s - h^t\|_{L^p(\mathbb{R}^n)} + \|h^t - f^t\|_{L^p(\mathbb{R}^n)} \\ &= \|(f-h)^s\|_{L^p(\mathbb{R}^n)} + \|h^s - h^t\|_{L^p(\mathbb{R}^n)} + \|(h-f)^t\|_{L^p(\mathbb{R}^n)} \\ &= \|f - h\|_{L^p(\mathbb{R}^n)} + \|h^s - h^t\|_{L^p(\mathbb{R}^n} + \|g - f\|_{L^p(\mathbb{R}^n} \\ &< \delta.\end{aligned}$$

Therefore we have that the function $g(y)$ is continuous.

Having established boundedness and continuity of $g(y)$ we proceed by computing as follows

$$
\begin{aligned}
\|f * \phi_\varepsilon - f\|_{L^p(\mathbb{R}^n)}^p &\leq \int\limits_{\mathbb{R}^n} g(y)\phi_\varepsilon(y)dy \\
&= \int \mathbb{R}^n g(y)\varepsilon^{-n}\phi(\frac{y}{\varepsilon})dy, \\
&= \int\limits_{\mathbb{R}^n} g(\varepsilon s)\phi(s)ds,
\end{aligned}
$$

where the last line follows a simple change of variable $s = y\varepsilon^{-1}$.

As $\varepsilon \to 0$ we have $g(\varepsilon s)\phi(s) \to g(0)\phi(s) = 0$ pointwise on $\mathbb{R}^n$. Moreover we can establish the upper bound $g(\varepsilon s)\phi(s) \leq \sup(g)\phi(s)$ for all ε. Since g is bounded and ϕ integrable we can use the Dominated Convergence Theorem to finish off the proof by showing that

$$
\lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^n} g(\varepsilon)\phi(s)ds = 0.
$$

This implies that $\|f * \phi_\varepsilon - f\|_{L^p(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$ and thus completes the proof.

□

Remark 3. Theorem 13 also holds for any function ϕ_ε that is nonnegative, measurable on $\mathbb{R}^n$ and of total integral one.

Theorem 14. *If Ω is an open set in $\mathbb{R}^n$ and K a compact subset of Ω, then there exists a function $\psi \in C_c^\infty(\Omega)$ with $0 \leq \psi \leq 1$ such that $\psi = 1$ in a neighborhood of K.*

Proof. Let $\varepsilon > 0$ be sufficiently small such that $\|x - y\| \geq 4\varepsilon$ for all $x \in K$ and $y \in \partial\Omega$. Let v be the characteristic function of $K_{2\varepsilon} = \{y \in \Omega \mid \|x - y\| \leq 2\varepsilon\ \}$. Now we define the function ψ as follows

$$
\psi = v * \phi_\varepsilon \in C_c^\infty(K_{3\varepsilon}).
$$

C^∞ regularity of ψ follows from Theorem 9 and compactness follows from Lemma 10 since $\mathrm{supp}(\psi) \subset \mathrm{supp}(v) + \overline{B_\varepsilon(0)} \subset K_{3\varepsilon}$.

To finish the proof we just show that $1 - \psi = (1 - v)\phi_\varepsilon$ vanishes in K_ε, which follows directly

$$
\mathrm{supp}(1 - \psi) \subset \mathrm{supp}(1 - v) + \overline{B_\varepsilon} = \Omega \backslash K_{2\varepsilon} + B_\varepsilon = \Omega \backslash K_\varepsilon.
$$

□

We are now ready to show the theorem that motivated the creation of this document. We recall the theorem and then proceed to prove it

Theorem 15. *Given an open set $\Omega \subset \mathbb{R}^n$ and a function $f \in L^1_{\mathrm{loc}}(\Omega)$ we have that if $\int\limits_\Omega f\varphi dx = 0$ for all $\varphi \in C_0^\infty(\Omega)$, then $f = 0$ almost everywhere in Ω.*

Proof. Let $K \subset\subset \Omega$ be a compact subser of Ω. By Theorem 14 we know that there exists a function $\psi \in C_c^\infty(\Omega)$ such that $\psi = 1$ on a neighbourhood of K.

Let $f_\psi = f\psi$, which means that $f_\psi \in L^1(\mathbb{R}^n)$, since f is $L^1_{\mathrm{loc}}(\Omega)$ and $\psi \in C_c^\infty(\Omega)$. Take any mollifier, such as the standard mollifier $\phi \in C_c^\infty(\mathbb{R}^n)$.

For a fixed x we have $y \mapsto \psi(y)\phi_\varepsilon(x - y)$ is of $C_c^\infty(\Omega)$ regularity and thus by hypothesis on f we have for any $x \in \mathbb{R}^n$

$$
(\phi_\varepsilon * f_\psi)(x) = \int\limits_\Omega f(y)\psi(y)\phi_\varepsilon(x - y)dy = 0. \tag{1}
$$

By Theorem 13 we know that

$$\phi_\varepsilon * f_\psi \to f_\psi \text{ in } L^1(\mathbb{R}^n). \tag{2}$$

Combining 1 and 2 we can conclude that

$$\lim_{\varepsilon\to 0}(\phi_\varepsilon * f_\psi)(x) = f_\psi(x) = 0 \text{ a.e. in } \mathbb{R}^n.$$

Since $f_\psi(x) = f(x)\psi(x) = f(x)$ on K since $\psi(x) = 1$ on K we conclude that

$$f(x) = 0 \text{ a.e. on } K,$$

where K is any compact subset of Ω and thus we have $f = 0$ a.e. in Ω, which concludes the proof. □

Bibliography

[1] Walter Rudin. *Real and complex analysis, Third Edition*. McGraw-Hill, 1987.