Products of CW complexes

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- Basics of CW Complexes
- Products of CW Complexes
- Dowker's example
- History of characterization results for products of CW Complexes
- A complete characterization without set theoretic assumptions

In algebraic topology, most topological spaces are not easy to work with. Even working with spheres is not as straightforward as one may think.

That's where CW complexes come in very handy:

Spaces constructed by gluing n-disks of various dimensions.

Notation

We will denote the closed unit ball in $\mathbb{R}^n$ by D^n, its interior by E^n and its boundary, i.e. the $(n-1)$-sphere, by S^{n-1}.

Definition 1: CW Complex

A Hausdorff space X is a CW complex if there exist continuous functions $\varphi^n_\alpha : D^n \to X$ for α in an arbitary index set and $n \in \mathbb{N}$ a function of α, such that the following conditions hold:

1. The restriction $\varphi^n_\alpha|_{E^n}$ is a homeomorphism from E^n to $\operatorname{img} \varphi^n_\alpha|_{E^n} =: e^n_\alpha$.
2. X is the disjoint union of all e^n_α, each of which we call an n-dimensional cell.
3. For each φ^n_α, $\varphi^n_\alpha(S^{n-1})$ is contained in finitely many cells, all of which are of dimension less than n.
4. The topology on X is the weak topology, i.e. a set is closed if and only if its intersection with each closed cell $\varphi^n_\alpha(D^n)$ is closed.

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A Hausdorff space X satisfying the first three conditions is a CW-complex if and only if X is sequential.

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3. X^2 is defined by taking copies of D^2 and attaching the boundary of D^2, i.e. S^1 to X^1.
4. Repeat inductively up to some finite n to get an n-dimensional CW complex. If we don't stop at some finite n, we get an infinite-dimensional CW Complex.

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- Brown's representability theorem

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But
Does the above topology coincide with the product topology?

Wikipedia: CW Complex

The product of two CW complexes can be made into a CW complex. Specifically, if X and Y are CW complexes, then one can form a CW complex $X \times Y$ in which each cell is a product of a cell in X and a cell in Y, endowed with the weak topology. The underlying set of $X \times Y$ is then the Cartesian product of X and Y, as expected. In addition, the weak topology on this set **often agrees** with the more familiar product topology on $X \times Y$.

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As we saw previously, any product of two CW complexes can be given a cell structure along with the weak topology such that we end up with a CW complex $X \times Y$.

1952, Dowker

First example of the product topology on $X \times Y$ of two CW-complexes differing from the CW topology

Construction of Dowker's example

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Let $X = \bigvee_k I_k$, where I_k is a copy of the interval $[0, 1]$ and k ranges over all infinite sequences $k = (k_1, k_2, ...)$ of positive integers. The wedge sum is formed at the endpoint 0 of I_k.

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We do the same for Y, except that instead we take the wedge sum over positive integers.

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P is closed in the CW topology on $X \times Y$.

Our goal is to show that P is not closed in the product topology.
To do so, we show that some (x, y) is in the closure of each 2-cell,
with x being the common endpoint of the intervals I_k
and y being the common endpoints of the intervals I_j.

Construction of Dowker's example

Take a basic open set containing (x, y) in the product topology.
Such a set is of the form $U \times V$, where $U = \bigvee_k [0, a_k)$ and $V = \bigvee_j [0, b_j)$.

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Then $t_l > l > 1/a_t$ and hence $1/t_l < a_t$. Moreover, $1/t_l < b_l$. So $(1/t_l, 1/t_l)$ is a point of P that lies in $[0, a_t) \times [0, b_l)$ and therefore this point is in $U \times V$.

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3. For X a CW complex and $e_{X,\alpha}$ a cell of X, we denote by $X_\alpha^{\min}$ the minimal (with respect to inclusion) subcomplex of X containing e_α.

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X^n is an example of a subcomplex.

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Definition 4: Eventual domination

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If this is the case, we write $f \leq^* g$.

Definition 5: Bounding number $\mathfrak{b}$

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The bounding number $\mathfrak{b}$ is the least cardinality of a set of functions $\mathbb{N} \to \mathbb{N}$ that is unbounded with respect to eventual domination, i.e.

$$\mathfrak{b} := \min\{|\mathcal{F}| \ : \ \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \text{ and } \forall g \in \mathbb{N}^{\mathbb{N}} \ \exists f \in \mathcal{F} \text{ such that } \neg(f \leq^* g)\}$$

Definition 6: Singular and Regular

A cardinal κ is called singular if it can be expressed as follows:

$$\kappa = \bigcup_{\alpha<\gamma} I_\alpha,$$

with $\gamma < \kappa$ and $|I_\alpha| < \kappa$ for each $\alpha < \gamma$.

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If κ is not singular, we call it regular.

Lemma 1

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Proof

To show this, we take X to be the set of functions from $\mathbb{N}$ to $\mathbb{N}$ of cardinality $\mathfrak{b}$ which is unbounded with respect to eventual domination. We enumerate $X = \{f_\beta : \beta \in \mathfrak{b}\}$.

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Then for each α there must be some function $g_\alpha : \mathbb{N} \to \mathbb{N}$ that eventually dominates each member of $\{f_\beta : \beta \in I_\alpha\}$.

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Then for each α there must be some function $g_\alpha : \mathbb{N} \to \mathbb{N}$ that eventually dominates each member of $\{f_\beta : \beta \in I_\alpha\}$.

But then $\{g_\alpha : \alpha < \gamma\}$ would be an unbounded set of functions of cardinality $\gamma < \mathfrak{b}$, therefore contradicting the minimality of $\mathfrak{b}$.

Basic properties of $\mathfrak{b}$

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- If the Continuum hypothesis holds, then we have $\aleph_1 = \mathfrak{b} = 2^{\aleph_0}$.
- There exist models of set theory in which $\aleph_1 = \mathfrak{b} < 2^{\aleph_0}$, models in which $\aleph_1 < \mathfrak{b} = 2^{\aleph_0}$ and models in which $\aleph_1 < \mathfrak{b} < 2^{\aleph_0}$.

Theorem 1: (1949) Whitehead

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He added that he was unsure whether this condition was strictly necessary.

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Dowker's construction shows that some restrictions need to be put on X and/or Y, but it turns out that we can weaken the condition that either X or Y is locally finite.

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If X and Y have countably many cells, then $X \times Y$ is a CW complex.

Theorem 3: (1982) Tanaka

If neither X nor Y is locally countable, then $X \times Y$ is not a CW complex.

Theorem 4: (1978) Ying-Ming

Assuming the Continuum Hypothesis, $X \times Y$ is a CW complex if and only if one of them is locally finite, or both are locally countable.

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Theorem 5: (1982) Tanaka

Assuming $\mathfrak{b} = \aleph_1$, $X \times Y$ is a CW complex if and only if one of them is locally finite, or both are locally countable.

Complete Characterization: (2017) Brooke-Taylor

Let X and Y be CW complexes. Then $X \times Y$ is a CW complex if and only if one of the following holds:

1. Either X or Y is locally finite.
2. Either X or Y has countable many cells in each connected component, and the other has fewer than $\mathfrak{b}$ many cells in each connected component.

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For the converse, more work is required. Let κ be an uncountable regular cardinal and suppose that X is a locally less than κ CW complex.
Let $x \in X$ be some point.

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Let $x \in X$ be some point.

We want to show that the connected component of X containing x contains fewer than κ many cells.
This can be done by a recursive construction of the aforementioned component.

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Let $A_0 := \emptyset$ and A_1 be a connected subcomplex of X containing x in its interior and consisting of fewer than κ many cells.

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Let $A_0 := \emptyset$ and A_1 be a connected subcomplex of X containing x in its interior and consisting of fewer than κ many cells.

Suppose that we defined a subcomplex A_i of X with fewer than κ many cells, containing x in its interior and with the property that every element of A_{i-1} is contained in the interior of A_i. Clearly, this holds for A_1.

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Let us consider a cell e of A_i. Since X is locally less than κ, for each $y \in \overline{e}$ there exists a connected subcomplex A_y of X with fewer than κ many cells alongside an open set $U_y \subset X$ such that $y \in U_y \subseteq A_y$.

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Let κ be an uncountable regular cardinal. Then a CW complex X is locally less than κ if and only if each connected component of X contains fewer than κ many cells.

Proof

It is obvious that $U_y \cap \overline{e}$ is open and for $z \in U_y \cap \overline{e}$ we know that z is in the interior of A_y. Since $\overline{e}$ is compact, a finite set S_e of points y suffices to cover $\overline{e}$ by sets $U_y \cap \overline{e}$.

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We define A_{i+1} as follows:

$$A_{i+1} := \bigcup_{\substack{e \text{ a cell} \\ \text{of } A_i}} \bigcup_{y \in S_e} A_y.$$

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Proof

Since each A_y has fewer than κ many cells, and the union is over fewer than κ many indices, by virtue of regularity of κ, A_{i+1} has fewer than κ many cells. Each A_y in the union is connected to A_i, so A_{i+1} is connected. By construction A_i is contained in the interior of A_{i+1}, which means we have completed the inductive step.

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To finish the proof, we define $A = \bigcup_{i \in \mathbb{N}} A_i$. Since κ is regular uncountable and each A_i has fewer than κ many cells, A has fewer than κ many cells. An increasing union of connected spaces is connected, which means A is connected. Additionally, by construction A is open and as a subcomplex of X it is closed, so A is clearly a connected component of X.

Whitehead: 1949

CW complexes are normal (i.e. T4 spaces).

Before we show this result, we introduce a convenient way of constructing open neighborhoods $N_\varepsilon(A)$ of subsets A of a CW complex X, where ε is a function assigning a number $\varepsilon_\alpha > 0$ to each cell e^n_α of X.

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Then we define $N^{n+1}_\varepsilon(A)$ by specifying its preimage under the characteristic map $\varphi_\alpha : D^{n+1} \to X$ of each cell e^{n+1}_α, namely $\varphi^{-1}_\alpha(N^{n+1}_\varepsilon(A))$ is the union of two parts: an open ε_α-neighbourhood of $\varphi^{-1}_\alpha(A) \setminus \partial D^{n+1}$ in $D^{n+1} \setminus \partial D^{n+1}$, and a product $(1 - \varepsilon_\alpha, 1] \times \varphi^{-1}_\alpha(N^n_\varepsilon(A))$ with respect to 'spherical' coordinates (r, θ) in D^{n+1}, where $r \in [0, 1]$ is the radial coordinate and θ lies in $\partial D^{n+1} = S^n$.

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Then we define $N_\varepsilon(A) = \bigcup_n N^n_\varepsilon(A)$. This is an open set in X since it pulls back to an open set under each characteristic map.

Proof: CW complexes are normal

Points are closed in a CW complex X since they pull back to closed sets under all characteristic maps φ_α. For disjoint closed sets A and B in X, we show that $N_\varepsilon(B)$ are disjoint for small enough ε_α's. In the inductive process for building these open sets, assume $N^n_\varepsilon(A)$ and $N^n_\varepsilon(B)$ have been chosen to be disjoint.

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For a characteristic map $\varphi_\alpha : D^{n+1} \to X$, observe that $\varphi_\alpha^{-1}(N^n_\varepsilon(A))$ and $\varphi_\alpha^{-1}(B)$ are a positive distance apart, since otherwise by compactness we would have a sequence in $\varphi_\alpha^{-1}(B)$ converging to a point of $\varphi_\alpha^{-1}(B)$ in ∂D^{n+1} of distance zero from $\varphi_\alpha^{-1}(N^n_\varepsilon(A))$, but this is impossible since $\varphi_\alpha^{-1}(N^n_\varepsilon(B))$ is a neighborhood of $\varphi_\alpha^{-1}(B) \cap \partial D^{n+1}$ in ∂D^{n+1} disjoint from $\varphi_\alpha^{-1}(N^n_\varepsilon(A))$.

Proof: CW complexes are normal

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Similarly, $\varphi_\alpha^{-1}(N_\varepsilon^n(B))$ and $\varphi_\alpha^{-1}(A)$ are a positive distance apart. Also, $\varphi_\alpha^{-1}(A)$ and $\varphi_\alpha^{-1}(B)$ are a positive distance apart. So a small enough ε_α will make $\varphi_\alpha^{-1}(N_\varepsilon^{n+1}(A))$ disjoint from $\varphi_\alpha^{-1}(N_\varepsilon^{n+1}(B))$ in D^{n+1}.

Brooke-Taylor: 2017

Let X and Y be CW complexes. Then $X \times Y$ is a CW complex if and only if one of the following holds:

1. X or Y is locally finite.
2. One of X and Y is locally countable, and the other is locally less than $\mathfrak{b}$.

Proof

One direction follows directly from a theorem of Tanaka:

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Proof

One direction follows directly from a theorem of Tanaka:

Tanaka

The following are equivalent

1. $\kappa \geq \mathfrak{b}$,
2. If $X \times Y$ is a CW complex, then either
   - X or Y is locally finite, or
   - X or Y is locally countable and the other is locally less than κ.

Thanks to Proposition 1, we only need to show that if $\kappa = \mathfrak{b}$, we have that either of the following two conditions

- X or Y is locally finite, or
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implies that $X \times Y$ is a CW complex.

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implies that $X \times Y$ is a CW complex.

The first of the two conditions clearly implies that $X \times Y$ is a CW complex. This is the original result shown by Whitehead that we saw earlier.

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We follow the standard notation from set theory, that when a natural number n is used in place of a set of natural numbers, it denotes the n-element set $\{0, ..., n-1\}$.

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We follow the standard notation from set theory, that when a natural number n is used in place of a set of natural numbers, it denotes the n-element set $\{0, ..., n-1\}$.

For a function $s : I \to K$, the function that extends s by taking value q on some $\alpha \notin I$ is denoted by $s \cup \{(\alpha, q)\}$. We start by defining a descending sequence of neighbourhoods $B_n(x)$ open in a cell e that form a neighbourhood base in e of a point x.

Definition 7

Suppose x is a point in a CW complex X, with x lying in an open cell e of dimension d with characteristic map φ, and suppose n is a natural number.

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Let z be $\varphi^{-1}(x)$, and let $r \in \mathbb{R}$ be the minimum of $1/(n+1)$ and half the distance from z to the boundary of D^d. Then we define $B_n(x)$ to be the image under φ of the open ball of radius r about z in D^d.

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The set $B_n(x)$ need not be open as a subset of X. To build an open neighbourhood in X we must also consider higher-dimensional cells whose boundaries intersect $B_n(x)$. For these cells we use the following "collar neighbourhoods":

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Let e be a $(d+1)$-dimensional cell of X with characteristic map φ, and let n be a natural number. We define the open subset $C_n^e(U)$ of $\overline{e}$ by

$$C_n^e(U) = \varphi\Big(\{t \cdot z \ : \ t \in (\frac{n}{n+1}, 1] \text{ and } z \in \varphi^{-1}(U) \subseteq S^d\}\Big)$$

where the $\cdot$ denotes scalar multiplication in the vector space $\mathbb{R}^{d+1}$.

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Note that if $\varphi^{-1}(U)$ is empty then $C_n^e(U)$ will also be empty, and that C_n^e distributes over unions: for any U and V, $C_n^e(U \cup V) = C_n^e(U) \cup C_n^e(V)$.

Definition 9

Suppose X is a CW complex with its cells enumerated as e_i for i in some index set I, and for each i in I let $d(i)$ be the dimension of e_i. Then for each $n \in \mathbb{N}$ we let $I^n = \{i \in I \ : \ d(i) \leq n\}$.

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Thus, for finite n the n-skeleton X^n is the union over i in I^n of the cells e_i. Using these notions, we may define an open neighbourhood of a point from a function to the naturals.

Definition 10

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Then for any function $f\colon I \to \mathbb{N}$ we define the open neighbourhood $U^X(x; f)$, or simply $U(x; f)$ when X is clear, of x in X recursively in dimension as follows.

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- For all i in $I^{d(i_0)}$ other than i_0 , we take $U^X(x; f) \cap e_i = \emptyset$.
- For $i = i_0$, we take $U^X(x; f) \cap e_i = B_{f(i)}(x)$.
- If $U^X(x; f) \cap X^m$ has been defined for some $m \geq d(i_0)$, and $i \in I$ is such that $d(i) = m + 1$, we set $U^X(x; f) \cap \overline{e_i} = C^{e_i}_{f(i)}(U^X(x; f) \cap X^m)$.

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Clearly every such set $U^X(x; f)$ is open in X.

Note also that if A is a subcomplex of X and $J \subseteq I$ is the set of indices of cells in A, $J = i \in I \ : \ e_i \subseteq A$, then $U^A(x; f|_J) = U^X(x; f) \cap A$.

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Also, as per the set-theoretic convention discussed above, $f \uparrow i$ denotes the restriction of f to natural numbers less than i, $f \uparrow i = f|_{\{0,\ldots,i-1\}}$.

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Since each $U(x; f)$ for $f \colon I \to \mathbb{N}$ is open, it will suffice for our proof of Theorem 1 to produce sets of this form. In some sense this is also necessary:

Lemma 2

For any CW complex X with cells e_i , $i \in I$, and for any x in X, the sets $U(x; f)$ as f varies over functions from I to $\mathbb{N}$ form an open neighbourhood base at x.

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Proof

Given an open neighbourhood V of x, we construct recursively on dimension a function $f : I \to \mathbb{N}$ such that $U(x; f) \cap X^n \subset V \cap X^n$ for every $n \in \mathbb{N}$.

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If x is in cell e_{i_0} of dimension $d(i_0)$, then as the base case we may choose $f(i_0)$ large enough that $B_{f(i_0)}(x)$ has closure contained in V , since $V \cap e_{i_0}$ is open in e_{i_0}, and set $f(i) = 0$ for every other i in $I_{d(i_0)}$.

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Proof

For the inductive step, suppose we have defined f on I^n in such a way that $U(x; f \downarrow n) \subset V \cap X^n$, and suppose e_l is an $(n+1)$-cell of X with characteristic map φ_l.

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Then $\varphi_l^{-1}(U(x; f \downarrow n))$ is a compact subset of $\varphi_l^{-1}(V) \cap S^n$, and thus we may choose $f(l)$sufficiently large that $C^{e_l}_{f(l)}(U(x; f \downarrow n))$ also has closure contained in $\varphi_l^{-1}(V)$.

We shall repeatedly require the following lemma allowing us to extend open sets on finite subcomplexes.

Lemma 3

Suppose W and Z are CW complexes, $\tilde{W}$ is a finite subcomplex of W, $\tilde{Z}$ is a finite subcomplex of Z, U is a subset of $\tilde{W}$ that is open in $\tilde{W}$, V is a subset of $\tilde{Z}$ that is open in $\tilde{Z}$, and H is a sequentially closed subset of $W \times Z$ such that the closure of $U \times V$ is disjoint from H.

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Let e be a cell of Z whose boundary is contained in $\tilde{Z}$.

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Then there is a $p \in \mathbb{N}$ such that $U \times (V \cup C_p^e(V))$ has closure disjoint from H.

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The point is that $V \cup C_p^e(V)$ is open in $\tilde{Z} \cup e$, and we can build up open sets in the full CW complex Z in this way.

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The point is that $V \cup C_p^e(V)$ is open in $\tilde{Z} \cup e$, and we can build up open sets in the full CW complex Z in this way.

Note also that apart from which CW complex e belongs to, Lemma 3 is symmetric in W and Z, so we will be able to use it to build up open sets of both X and Y in the proof of the main theorem.

Proof

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Thus $H \cap (\tilde{W} \times \tilde{Z}e)$ is a closed subset of $\tilde{W} \times \tilde{Z}e$ disjoint from $\overline{U \times V}$, and so we may take disjoint open sets $O_{U \times V}$ and O_H in $\tilde{W} \times \tilde{Z}e$ such that $\overline{U \times V} \subseteq O_{U \times V}$ and $H \cap (\tilde{W} \times \tilde{Z}e) \subseteq O_H$.

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Now, for every point (u, v) of $U \times V$, there is an open base set $R \times S$ of the product topology on $\tilde{W} \times \tilde{Z}e$ that contains (u, v) and is contained in $O_{U \times V}$.

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By shrinking S if necessary, we may assume S is of the form $T \cup C_n^e(T)$ for some open subset T of $\tilde{Z}$ and some $n \in \mathbb{N}$ (recall that this also makes sense if $T \cap \overline{e}$ is empty, in which case n is arbitrary).

Proof

Now, by compactness of $U \times V$, finitely many such base sets $R \times S$ suffice to cover $U \times V$, and we may choose $p \in \mathbb{N}$ to be strictly greater than all of the corresponding values n.

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Then $U \times (V \cup C_p^e(V))$ has closure contained in $O_{U \times V}$, and hence disjoint from H, as required.

We return to the main theorem we want to prove. By proposition 1, the formulation given here is equivalent to the main theorem given earlier.

Theorem

Let X and Y be CW complexes. Then $X \times Y$ is a CW complex if and only if one of the following holds:

- X or Y is locally finite.
- One of X and Y is locally countable, and the other is locally less than $\mathfrak{b}$.

As discussed previously, it suffices to show that if X has countably many cells and Y has fewer than $\mathfrak{b}$ many cells, then $X \times Y$ is a CW complex.

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We shall show that the product topology on $X \times Y$ is sequential, and so indeed makes $X \times Y$ a CW complex.

To this end, let H be an arbitrary sequentially closed subset of $X \times Y$, and take $(x_0, y_0) \in X \times Y \setminus H$.

As discussed previously, it suffices to show that if X has countably many cells and Y has fewer than $\mathfrak{b}$ many cells, then $X \times Y$ is a CW complex.

So suppose X is a CW complex with countably many cells and Y is a CW complex with fewer than $\mathfrak{b}$ many cells.

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To this end, let H be an arbitrary sequentially closed subset of $X \times Y$, and take $(x_0, y_0) \in X \times Y \setminus H$.

We want to construct an open neighbourhood of (x_0, y_0) disjoint from H.

Enumerate the cells of X as $e_{X,i}$ for i in $\mathbb{N}$, in such a way that for each i, the boundary of $e_{X,i}$ is contained in $\bigcup_{j<i} e_{X,j}$. This is possible by closure-finiteness.

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We define the finite subcomplex X_i of X to be $X_i = \bigcup_{j\leq i} e_{X,j}$.

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Enumerate the cells of Y as $e_{Y,\alpha}$ for α in some index set J with cardinality $\mu < \mathfrak{b}$ (we leave J abstract rather than declaring $J = \mu$ so that the notation J^n of Definition 9 remains clear).

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Recall our notation $Y_\alpha^{\min}$ from Definition 2 for the minimal subcomplex of Y containing $e_{Y,\alpha}$.

Let $m(i)$ be the dimension of cell $e_{X,i}$, and let $n(\alpha)$ be the dimension of cell $e_{Y,\alpha}$. Let e_{X,i_0} be the unique open cell of X containing x_0, and e_{Y,α_0} the unique open cell of Y containing y_0.

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As ever, the construction is by recursion, but we shall recurse over dimension on the Y side and over i on the X side, whilst also keeping track of a lower bound function for the X side.

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As ever, the construction is by recursion, but we shall recurse over dimension on the Y side and over i on the X side, whilst also keeping track of a lower bound function for the X side.

Specifically, we shall construct for each i in $\mathbb{N}$ functions $f_i : \mathbb{N} \to \mathbb{N}$ and $g_i : J^{n(\alpha_0)+i} \to \mathbb{N}$ such that

- $U(x_0; f_i) \times U(y_0; g_i)$ has closure disjoint from H,
- for all $j > i$, $g_j|_{I_n}(\alpha_0) + i = g_i, f_j \uparrow i = f_i \uparrow i$, and for all $n \geq i$, $f_j(n) \geq f_i(n)$.

With such functions in hand we may define f and g by $f(i) = f_{i+1}(i)$ and $g(\alpha) = g_{n(\alpha)-n(\alpha_0)}(\alpha)$.

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Then $U(x_0; f) \times U(y_0; g) = \bigcup_{i \in \mathbb{N}} U(x_0; f \uparrow i) \times U(y_0; g \downarrow n(\alpha_0) + i) = \bigcup_{i \in \mathbb{N}} U(x_0; f_i \uparrow i) \times U(y_0; g_i)$, each term of which will be disjoint from H by construction.

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For the base case of the construction, consider $X \times Y^{\min}_{\alpha_0}$. Since $Y^{\min}_{\alpha_0}$ is a finite CW complex, $X \times Y^{\min}_{\alpha_0}$ is a CW complex, $(X \times Y^{\min}_{\alpha_0}) \cap H$ is closed, and we may choose a function $f_0 : \mathbb{N} \to \mathbb{N}$ and a natural number $g_0(\alpha_0)$ such that $U(x_0; f_0) \times B_{g_0(\alpha_0)}(y_0)$ has closure disjoint from H.

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For $\alpha \neq \alpha_0$ in $J^{n(\alpha_0)}$, set $g_0(\alpha) = 0$, so we have g_0 defined on all of $J^{n(\alpha_0)}$; since $U(y_0; g_0) = B_{g_0(\alpha_0)}(y_0)$, we have that $U(x_0; f_0) \times U(y_0; g_0)$ has closure disjoint from H.

Lemma 4

Let $\tilde{Y}$ be a finite subcomplex of Y containing y_0, let F be a function from $\mathbb{N}$ to $\mathbb{N}$ and s a function from the indices of $\tilde{Y}$ to $\mathbb{N}$ such that $U(x_0; F) \times U(y_0; s) \subseteq X \times \tilde{Y}$ has closure disjoint from H.

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Let i be a natural number and let $\hat{Y}$ be a subcomplex of Y that is a one cell extension of $\tilde{Y}$, $\hat{Y} = \tilde{Y} \cup e_\alpha$.

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Let i be a natural number and let $\hat{Y}$ be a subcomplex of Y that is a one cell extension of $\tilde{Y}$, $\hat{Y} = \tilde{Y} \cup e_\alpha$.

Then there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that

1. $f(n) \geq F(n)$ for all n in $\mathbb{N}$, and $f(n) = F(n)$ for all $n < i$,
2. for every $\tilde{f}\colon \mathbb{N} \to \mathbb{N}$ such that $\tilde{f} \geq f$ and $\tilde{f} \geq F$, there is a $q \in \mathbb{N}$ such that $U(x_0; \tilde{f}) \times U(y_0; s \cup \{(\alpha, q)\})$ has closure disjoint from H.

Proof

The construction of f is by recursion on $n \geq i$, with repeated applications of Lemma 3.

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For every sequence $r : n \to \mathbb{N}$ such that $F(m) \leq r(m) \leq f(m)$ for all $m < n$, let $q(r)$ be the least $q \in \mathbb{N}$ such that $U(x_0; r) \times U(y_0; s \cup \{(\alpha, q)\})$ has closure disjoint from H; such a q must exist by assumption on F and s and Lemma 3.

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Then let $p(r)$ be the least $p \in \mathbb{N}$ such that $U(x_0; r \cup \{(n, p)\}) \times U(y_0; s \cup \{(\alpha, q(r))\})$ has closure disjoint from H, again applying Lemma 3.

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Finally, we define $f(n)$ as $f(n) = \max(\{p(r) : F \uparrow n \leq r \leq f \uparrow n\} \cup F(n))$.

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We claim that this recursive construction yields a function $f : \mathbb{N} \to \mathbb{N}$ as per the statement of the lemma.

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Indeed we shall show by induction that, letting $\hat{f}$ be the function $\hat{f}(n) = \begin{cases} r(n) & \text{if } n < n_0, \\ f(n) & \text{if } n \geq n_0 \end{cases}$, we obtain that $U(x_0; \hat{f}) \times U(y_0; s \cup (\alpha, q(r)))$ has closure disjoint from H.

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The result will then follow, as $\tilde{f} \geq \hat{f}$ and hence $U(x_0; \tilde{f}) \subseteq U(x_0; \hat{f})$.

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Also $\hat{f}(n) = f(n) \geq p(\hat{f}\uparrow n)$; so $U(x_0; \hat{f}\uparrow n+1) \times U(y_0; s \cup \{(\alpha, q(\hat{f}\uparrow n))\})$ has closure disjoint from H, whence the possibly smaller set $U(x_0; \hat{f}\uparrow n+1) \times U(y_0; s \cup \{(\alpha, q(r))\})$ has closure disjoint from H, as required for the inductive step.

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We therefore have that for every n, $U(x_0; \hat{f} \uparrow n) \times U(y_0; s \cup \{(\alpha, q(r))\})$ has closure disjoint from H.

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We therefore have that for every n, $U(x_0; \hat{f}\uparrow n) \times U(y_0; s \cup \{(\alpha, q(r))\})$ has closure disjoint from H.

Since $\bigcup_{n \in \mathbb{N}} U(x_0; \hat{f}\uparrow n) \times U(y_0; s \cup \{(\alpha, q(r))\})$ is closed in every cell of $X \times \hat{Y}$, it is closed in $X \times \hat{Y}$, and so $U(x_0; \hat{f}) \times U(y_0; s \cup \{(\alpha, q(r))\})$ has closure disjoint from H, as required.

Returning to the construction of the functions f_i and g_i for i in $\mathbb{N}$, suppose that for all $j \leq k$ we have constructed the functions $f_j : \mathbb{N} \to \mathbb{N}$ and $g_j : J^{n(\alpha_0)+j} \to \mathbb{N}$ satisfying the previously listed requirements.

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For relevant α, let $\tilde{Y}_\alpha$ be $(Y_{\alpha_0}^{\min} \cup Y_{\alpha}^{\min}) \setminus e_\alpha$ and let J_α be the set of indices of cells in $\tilde{Y}_\alpha$.

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Apply the Lemma 4 with f_k as F, $Y^{\min}_{\alpha_0} \cup Y^{\min}_{\alpha}$ as $\hat{Y}$, $\tilde{Y}_\alpha$ as $\tilde{Y}$, $g_k \uparrow J^\alpha$ as s, and $k+1$ as i.

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The requirement of Lemma 4 that $U(x_0; f_k) \times U(y_0; g_k \uparrow J_\alpha)$ have closure disjoint from H holds by the inductive hypothesis.

We thus get for each relevant α a function $f_{k+1,\alpha}$ satisfying (1) and (2) of Lemma 4.

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Since there are fewer than b many members of J, there is a single function $f_{k+1} : \mathbb{N} \to \mathbb{N}$ that eventually dominates $f_{k+1,\alpha}$ for every relevant α.

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Taking f_{k+1} as $\tilde{f}$ in (2) of Lemma 4, we have that for each relevant α there is $q_\alpha \in \mathbb{N}$ such that the open subset $U(x_0; f_{k+1}) \times U(y_0; (g_k \uparrow J_\alpha) \cup \{(\alpha, q_\alpha)\})$ of $X \times (Y_{\alpha_0}^{\min} \cup Y_\alpha^{\min})$ has closure disjoint from H

Products commute with closures in the product topology, therefore we have

$$\begin{aligned}&\overline{U(x_0; f_{k+1}) \times U(y_0; (g_k \uparrow J_\alpha) \cup \{(\alpha, q_\alpha)\})}\\ = &\overline{U(x_0; f_{k+1})} \times \overline{U(y_0; (g_k \uparrow J_\alpha) \cup \{(\alpha, q_\alpha)\})},\end{aligned}$$

and since $Y^{n(\alpha_0)+k+1}$ has the weak topology, we have

$$\begin{aligned}&\overline{\bigcup_{\alpha \text{ relevant}} U(y_0; (g_k \uparrow J_\alpha) \cup \{(\alpha, q_\alpha)\})}\\ = &\bigcup_{\alpha \text{ relevant}} \overline{U(y_0; (g_k \uparrow J_\alpha) \cup \{(\alpha, q_\alpha)\})}.\end{aligned}$$

So

$$U(x_0; f_{k+1}) \times \bigcup_{\alpha \text{ relevant}} U(y_0; (g_k \uparrow J_\alpha) \cup \{(\alpha, q_\alpha)\})$$

is an open subset of $X \times Y^{n(\alpha_0)+k+1}$ with closure disjoint from H.

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Since $f_{k+1} \geq f_k$, we have $U(x_0; f_{k+1}) \subseteq U(x_0; f_k)$, and we can conclude that $U(x_0; f_{k+1}) \times U(y_0; g_k)$ has closure disjoint from H.

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Taking

$$g_{k+1} = g_k \cup \{(\alpha, q_\alpha) : \alpha \text{ is relevant}\}$$

completes the inductive step.

We thus have a recursive construction of the functions f_i and g_i as required, which as discussed above allows us to form the functions f and g defining an open neighbourhood $U(x_0; f) \times U(y_0; g)$ of (x_0, y_0) disjoint from H.

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Since H was an arbitrary sequentially closed subset of $X \times Y$ and (x_0, y_0) was an arbitrary point in the complement of H in $X \times Y$, this shows that $X \times Y$ is sequential, and thus bears the weak topology.

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Therefore, $X \times Y$ is a CW complex.

References

- Andrew D. Brooke-Taylor. Products of CW complexes. arXiv:1710.05296 [math.GN], Oct. 2017.
- Allen Hatcher. Algebraic Topology. Cambridge University Press, Cambridge, UK, 2002.
- C. H. Dowker. Topology of metric complexes. American Journal of Mathematics, 74:557–577, 1952.
- J. H. C. Whitehead. Combinatorial homotopy I. Bulletin of the American Mathematical Society, 55(3):213–245, 1949.
- John Milnor. Construction of universal bundles, I. Annals of Mathematics, 63(2):272–284, March 1956.
- Yoshio Tanaka. Products of CW-complexes. Proceedings of the American Mathematical Society, 86(3):503–507, November 1982.
- Liu Ying-Ming. A necessary and sufficient condition for the products of CW-complexes. Acta Mathematica Sinica, 21:171–175, 1978. Chinese.