Tensor product of modules

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The material presented in this document follows the construction of the tensor product that can be found in [1]. Where deemed sensible, complementary remarks, clarifications, more precise arguments and illustrative examples are given by the author.

Remark 1. Unless stated otherwise, we consider A to be a commutative ring with multiplicative identity. The multiplicative identity 1 may be equal to 0. However, for such a ring the only bilinear map of form $M \times N \to P$, where M, N, P are A-modules, is the zero map and thus this case is of little interest to us.

Definition 1. Let M, N, P be three A-modules. A mapping $f : M \times N \to P$ is said to be A-bilinear if for each $x \in M$ the mapping $y \mapsto f(x, y)$ of N into P is A-linear, and for each $y \in N$ the mapping $x \mapsto f(x, y)$ of M into P is A-linear.

The idea of the tensor product of modules is to find a suitable A-module T, such that there exists a natural one-to-one correspondence between the A-bilinear mappings $M \times N \to T$ and the A-linear mappings $T \to P$, for all A-modules P. In less precise terms, we are trying to construct a A-module T that has the property of one-to-one correspondence between the A-linear mappings from T to P and the A-bilinear mappings from $M \times N$ to P, for any A-module P. Formally, this yields the following proposition.

Proposition 2. *Let M, N be A-modules. Then there exists a pair (T, g), where T is an A-module and g is an A-bilinear mapping $M \times N \to T$, for which the following holds:*

Given any A-module P and any A-bilinear mapping $f : M \times N \to P$, there exists a unique A-linear mapping $f' : T \to P$ such that $f = f' \circ g$.

Moreover, this pair is unique up to isomorphism, i.e. if (T, g) and (T', g') are two such pairs, then there exists a unique isomorphism $j : T \to T'$ such that $j \circ g = g'$.

Proof. **Uniqueness** Replacing (P, f) by (T', g') we get a unique $j : T \to T'$ such that $g' = j \circ g$. Interchanging the roles of T and T', we get $j' : T' \to T$ such that $g = j' \circ g'$. Each of the compositions $j \circ j'$ and $j' \circ j$ must be the identity, and therefore j is an isomorphism.

Existence Let C denote the free A-module $A^{(M \times N)}$. The elements of C are formal linear combinations of elements of $M \times N$ with coefficients in A, i.e. they are expressions of the form $\sum_{i=1}^{n} a_i \cdot (x_i, y_i)$, $(a_i \in A, x_i \in M, y_i \in N)$.

Let D be the submodule of C generated by all elements of C of the following types:

$$\begin{aligned} (x + x', y) - (x, y) - (x', y) \\ (x, y + y') - (x, y) - (x, y') \\ (ax, y) - a \cdot (x, y) \\ (x, ay) - a \cdot (x, y). \end{aligned} \tag{1}$$

Let $T = C/D$. For each basis element (x, y) of C, let $x \otimes y$ denote its image in T. Then T is generated by the elements of the form $x \otimes y$, and from our definitions we have

$$\begin{aligned} (x + x') \otimes y &= x \otimes y + x' \otimes y, \\ x \otimes (y + y') &= x \otimes y + x \otimes y', \\ (ax) \otimes y = x \otimes (ay) &= a(x \otimes y). \end{aligned} \tag{2}$$

Equivalently, the mapping $g : M \times N \to T$ defined by $g(x, y) = x \otimes y$ is A-bilinear.

Any map f of $M \times N$ into an A-module P extends by linearity to an A-module homomorphism $\overline{f} : C \to P$. Suppose in particular that f is A-bilinear. Then, from the definitions, $\overline{f}$ vanishes on all the generators of D, hence on the whole of D, and therefore induces a well-defined A-homomorphism f' of $T = C/D$ into P such that $f'(x \otimes y) = f(x, y)$. The mapping f' is uniquely defined by this condition, and therefore the pair (T, g) satisfy the conditions of the proposition. □

Remark 2. The module T in the above construction is the *tensor product* of M and N, and is denoted by $M \otimes_A N$, or just $M \otimes N$ if there happens to be no ambiguity about the ring A. It is generated as an A-module by the "products" $x \otimes y$. If $(x_i)_{i\in I}, (y_j)_{j\in J}$ are families of generators of M, N respectively, then the elements $x_i \otimes y_j$ generate $M \otimes N$. In particular, if M and N are finitely generated, so is $M \otimes N$.

Remark 3. The notation $x \otimes y$ is inherently ambiguous unless the tensor product to which it belongs is specified. In particular, given M', N' two submodules of M, N respectively and $x \in M', y \in N'$. Then we may have $x \otimes y$ is zero as an element of $M \otimes N$, but non-zero as an element of $M' \otimes N'$. The standard example for this case is $A = \mathbb{Z}$, $M = \mathbb{Z}$, $M' = 2\mathbb{Z}$, $N = N' = \mathbb{Z}/2\mathbb{Z}$. Take any non-zero element x of N. Consider $2 \otimes x$. As an element of $M \otimes N$, it is zero because $2 \otimes x = 1 \otimes 2x = 1 \otimes 0 = 0$. But as an element of $M' \otimes N'$ it is non-zero.

To avoid the aforementioned problem, we have the following result:

Corollary 3. *Let $x_i \in M$, $y_i \in N$ be such that $\sum x_i \times y_i = 0$ in $M \otimes N$. Then there exist finitely generated submodules M_0 of M and N_0 of N such that $\sum x_i \otimes y_i = 0$ in $M_0 \otimes N_0$.*

Proof. If $\sum x_i \otimes y_i = 0$ in $M \otimes N$, then using the notation introduced in Prop (2) we have $\sum(x_i, y_i) \in D$, and therefore $\sum(x_i, y_i)$ is a finite sum of generators of D. Let M_0 be the submodule of M generated by the x_i and all the elements of M which occur as first coordinates in these generators of D, and define N_0 analogously. Then $\sum x_i \otimes y_i = 0$ as an element of $M_0 \otimes N_0$. $\square$

Remark 4. The particulars of the construction of the tensor product introduced in Prop (2) serve no function beyond providing a reasonably comprehensible proof of the proposition. What really matters is the defining property of the tensor product, i.e. the statement provided in the propisition.

Remark 5. The construction of the tensor product as given above can be extended to the "multi-tensor product". Instead of starting with bilinear mappings, we start with multilinear mappings of the form $f : M_1 \times ... \times M_r \to P$ and end up with $T = M_1 \otimes ... \otimes M_r$, where T is generated by all products $x_1 \otimes ... \otimes x_r$ $(x_i \in M_i, 1 \leq i \leq r)$. This gives us the following result, which is given without proof.

Proposition 4. *Let $M_1, ..., M_r$ be A-modules. Then there exists a pair (T, g) consisting of an A-module T and an A-multilinear mapping $g : M_1 \times ... \times M_r \to T$, for which the following holds:*

Given any A-module P and any A-multilinear mapping $f : M_1 \times ... \times M_r \to T$, there exists a unique A-homomorphism $f' : T \to P$ such that $f' \circ g = f$.

Moreover, if (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism $j : T \to T'$ such that $j \circ g = g'$.

There exist various "canonical isomorphisms", some of which are given in the form of the following proposition.

Proposition 5 (Canonical isomorphisms)**.** *Let M, N, P be A-modules. There exist the following unique isomorphisms*

i) $M \otimes N \to N \otimes M$, *such that* $x \otimes y \mapsto y \otimes x$,

ii) $(M \otimes N) \otimes P \to M \otimes (N \otimes P) \to M \otimes N \otimes P$, *such that* $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z) \mapsto x \otimes y \otimes z$,

iii) $(M \oplus N) \otimes P \to (M \otimes P) \oplus (N \otimes P)$, *such that* $(x, y) \otimes z \mapsto (x \otimes z, y \otimes z)$,

iv) $A \otimes M \to M$, *such that* $a \otimes x \mapsto ax$.

Proof. For each isomorphism the required work consists of showing that the mappings as described in the proposition are well defined. This can be achieved by constructing suitable bilinear (respectively multilinear) mappings, and use the defininig property of the tensor product from Prop (2) or Prop (4) to infer the existence of homomorphisms of tensor products. We prove one half of the second isomorphism. The other isomorphisms can be constructed in a very similar fashion.

We set out to construct homomorphisms

$$(M \otimes N) \otimes P \xrightarrow{f} M \otimes N \otimes P \xrightarrow{g} (M \otimes N) \otimes P, \tag{3}$$

such that $f\big((x \otimes y) \otimes z\big) = x \otimes y \otimes z$ and $g\big(x \otimes y \otimes z\big) = (x \otimes y) \otimes z$ for all $x \in M$, $y \in M$, $z \in P$.

To construct f, fix an element $z \in P$. The mapping $(x, y) \mapsto x \otimes y \otimes z$ is bilinear in (x, y), which induces a homomorphism $f_z : M \otimes N \to M \otimes N \otimes P$ such that $f_z(x \otimes y) = x \otimes y \otimes z$.

Next, we consider the mapping $(M \otimes N) \times P \ni (t, z) \mapsto f_z(t) \in M \otimes N \otimes P$. This mapping is bilinear in (t, z) and induces another homomorphism

$$f : (M \otimes N) \otimes P \to M \otimes N \otimes P, \tag{4}$$

such that $f\big((x \otimes y) \otimes z\big) = x \otimes y \otimes z$.

To construct g, we consider the mapping $M \times N \times P \ni (x, y, z) \mapsto (x \otimes y) \otimes z \in (M \otimes N) \otimes P$. This mapping is linear in each variable and therefore induces a homomorphism

$$g : M \otimes N \otimes P \to (M \otimes N) \otimes P, \tag{5}$$

such that $g(x \otimes y \otimes z) = (x \otimes y) \otimes z$.

It is obvious that both $g \circ f$ and $f \circ g$ are identity maps, which means that f and g are isomorphisms. □

Proposition 6. *Let A, B be rings, M an A-module, P a B-module and N an (A, B)-bimodule, i.e. N is both an A-module and a B-module such that the two module structures are compatible in the sense that $a(xb) = (ax)b$ for all $a \in A, b \in B, x \in N$. Then $M \otimes_A N$ is naturally a B-module, $N \otimes_B P$ is an A-module and we have*

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P). \tag{6}$$

We can extend the tensor product to homorphisms of A-modules. This follows directly from elementary properties of the tensor product.

Let $f : M \to M'$, $g : N \to N'$ be two homomorphisms of A-modules. We define $h : M \times N \to M' \otimes N'$ by $h(x, y) = f(x) \otimes g(y)$. We can easily show that h is A-bilinear. For a fixed $x \in M$, the mapping $N \ni y \mapsto f(x) \otimes g(y)$ is linear since g is linear and the tensor product $\otimes$ is bilinear, which means it is linear with respect to $g(y)$. Similarly, for a fixed $y \in N$, the mapping $M \ni x \mapsto f(x) \otimes g(y)$ is linear.

The bilinearity of h induces an A-module homomorphism

$$f \otimes g : M \otimes N \to M' \otimes N', \text{ such that } (f \otimes g)(x \otimes y) = f(x) \otimes g(y) \quad (x \in M, y \in N). \tag{7}$$

Given two additional homomorphisms of A-modules $f' : M' \to M''$ and $g' : N' \to N''$, the following equality holds:

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g). \tag{8}$$

In fact, this is true because the two mappings $(f' \circ f) \otimes (g' \circ g)$ and $(f' \otimes g') \circ (f \otimes g)$ map $M \otimes N$ to $M'' \otimes N''$ and thus equality between them for all elements of the form $x \otimes y$ in $M \otimes N$ is enough for the equality to hold in general.

Let $f : M \times N \to P$ be an A-bilinear mapping. For each $x \in M$ the mapping $y \mapsto f(x, y)$ of N into P is A-linear, hence f gives rise to a mapping $M \to \operatorname{Hom}\big(N, P\big)$ which is A-linear because f is linear in the variable x. Conversely any A-homomorphism $\phi : M \to \operatorname{Hom}_A\big(N, P\big)$ defines a bilinear map, namely $(x, y) \mapsto \phi(x)(y)$. Hence the set S of A-bilinear mappings $M \times N \to P$ is in natural one-to-one correspondence with $\operatorname{Hom}\big(M, \operatorname{Hom}(N, P)\big)$, by the defining property of the tensor product. Hence we have a canonical isomorphism

$$\operatorname{Hom}\big(M \otimes N, P\big) \cong \operatorname{Hom}\big(M, \operatorname{Hom}(N, P)\big). \tag{9}$$

Proposition 7. *Let M, M', M'' be A-modules and u, v homomorphisms. Then, $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ exact if and only if for all A-modules N, the sequence*

$$0 \to \operatorname{Hom}(M'', N) \xrightarrow{\overline{v}} \operatorname{Hom}(M, N) \xrightarrow{\overline{u}} \operatorname{Hom}(M', N) \tag{10}$$

is exact.

Proof. See [1], Proposition 2.9. □

Proposition 8. *Let $E : M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of A-modules and homomorphisms, and let N be any A-module. Then the sequence*

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \to 0, \tag{11}$$

where 1 denotes the identity map on N, is exact.

Proof. By exacteness of E, using above proposition we deduce that for any A-module Q the following sequence is exact

$$0 \to \operatorname{Hom}(M'', Q) \to \operatorname{Hom}(M, Q) \to \operatorname{Hom}(M', Q). \tag{12}$$

Since for any A-module P, $\operatorname{Hom}(N, P)$ is an A-module, we can replace Q with $\operatorname{Hom}(N, P)$ and we have exacteness of the following sequence

$$0 \to \operatorname{Hom}(M'', \operatorname{Hom}(N, P)) \to \operatorname{Hom}(M, \operatorname{Hom}(N, P)) \to \operatorname{Hom}(M', \operatorname{Hom}(N, P)), \tag{13}$$

for any A-module P.

Using the above cannonical isomorphism, we replace each module in the sequence such that we get the following exact sequence

$$0 \to \operatorname{Hom}(M'' \otimes N, P) \to \operatorname{Hom}(M \otimes N, P) \to \operatorname{Hom}(M' \otimes N, P), \tag{14}$$

where P is any A-module.

Using the above proposition for a second time, we get that the sequence

$$M' \otimes N \to M \otimes N \to M' \otimes N, \tag{15}$$

is exact. □

We introduce two useful identities, which we will use in a concrete example later on. Let A be a commutative ring, I and J ideals and M, N A-modules. Then

$$\begin{aligned} A/I \otimes M &= M/IM, \\ A/I \otimes A/J &= A/(I + J). \end{aligned} \tag{16}$$

These identities are easy to verify. If we take the tensor product with M of the exact sequence $0 \to I \to A \to A/I \to 0$, we get

$$I \otimes M \xrightarrow{f} A \otimes M = M \to A/I \otimes M \to 0, \tag{17}$$

where $f(i \otimes x) := ix$. Since the image of f is exactly IM we deduce the first identity by exactness. For the second identity we have

$$A/I \otimes A/J = \frac{A/J}{I(A/J)} = \frac{A/J}{(I+J)/J} = A/(I+J), \tag{18}$$

where the first equality follows from the first identity.

Using the second identity, we have the following example.

$$\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m = \mathbb{Z}/\gcd(n,m) \tag{19}$$

Bibliography

[1] M. F. Atiyah; I. G. Macdonald. *Introduction to Commutative Algebra*. Addison-Wesley Publishing Company, 1969.